B W Rogers

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ADVANCED ALGEBRA

VOLUME II

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ADVANCED ALGEBRA

VOLUME II

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PREFACE

VOLUMES II and III complete the school course including suitable work for university scholars in their last terms at school.

The authors have attempted to concentrate attention on the fundamental principles, methods and notation which furnish the tools necessary for more advanced work. They believe that many of the topics which constitute the conventional course are only of value in so far as they illustrate general ideas, and that much of what has been called ‘ higher algebra ’ in the school course should be scrapped. Only the requirements of certain examinations have prevented them from pursuing a more drastic policy than they have actually adopted.

The account of the difference (Δ) notation in Chapter X forms an introduction to the study of difference equations in Chapter XI, and taken in conjunction with the sketch of the principles of probability in Chapter XVIII should enable those concerned with actuarial work to learn the essentials of these subjects before taking a specialist course. Some of the examples on probability may appear to be remote from actual life, but the more practical applications do not always provide the simplest illustrations of the principles involved. The philosophy of probability lies outside the scope of this work.

Difference equations, or recurrence formulae, are of great importance in mathematics, even apart from their valuable analogy with differential equations. Recurring series and continued fractions at least have the merit of providing illustrations of difference methods.

In Chapter XII the distinction between theorems of real and complex algebra is emphasised and for this purpose the authors believe that the new terminology introduced on p. 253 will be found of real service to the student: the mature mathematician may find it unnecessary. The main theme of this chapter is the fundamental theorem about $Af+Bg=1$, and special care has been taken to show how the theory of partial fractions can be derived from it. In practical decomposition into partial fractions the choice of the best method is a matter of experience and therefore the various alternatives have been copiously illustrated by examples in the text.

In Chapter XIII Descartes' Rule of Signs has been treated more fully than usual and its special value with incomplete equations has been emphasised. The importance of the considerations of weight and order in the theory of symmetric functions of the roots of an equation has been stressed. Newton's formula has been enunciated in a form slightly more comprehensive than is customary.

The early part of Chapter XIV is of great importance, because a sound understanding of the principles of convergence is essential; but the developments in the later part of the chapter should be left for a second reading. Although inequalities are not discussed systematically until Chapter XV, simple examples of their manipulation necessarily occur in Chapter XIV and the fundamental logarithmic inequality which was given on p. 108 of Volume I is required in some of the examples.

In Chapters XV, XVI, XVII the student is introduced to subjects of special significance in modern mathematics. Although he may be well-advised to rely at first on *ab initio* methods in dealing with inequalities, he can profitably make a start at learning the forms into which the simple special results can be generalised. To pursue this subject further he will naturally take up the study of *Inequalities* by Hardy, Littlewood, and Pólya. Attention is called to the introduction of the δ- and ϵ- symbols and the use of dummy suffixes. Too often the young student at the university is plunged into some subject in which these are the normal working tools, although he has had no preliminary training in their use. The same applies with even greater force to matrices.

The subject of Chapter XIX is a fascinating one. Any pure mathematician is certain to be attracted by it, even though the account here given does not give much indication of the lines of modern research in Theory of Numbers.

For the convenience of teachers, the exercises are divided into sections **A**, **B**, **C**: both the **A** and **B** questions are straightforward applications of the bookwork. It is suggested that all the **A** questions should be done. The **B** questions are intended for extra practice when this is necessary. The **C** questions are harder but have been carefully graded.

Short books of Hints for the solutions of any examples that are not immediate deductions from the bookwork have been compiled for Volumes II and III, and it is suggested that the student should have access to these books. Teachers cannot always find time to discuss various methods of handling a problem, and, even when the student has not failed to discover a solution, it will often be helpful to him to compare his method with another. The hints consist, in effect, of a very large number of illustrative examples solved in outline.

An index to Volumes I, II, III is given at the end of Volume III.

The thanks of the authors are due to Mr. W. Hope-Jones of Eton and Mr. T. A. A. Broadbent for advice on the probability and sequence chapters respectively, and to Mr. P. Hall of King's College and Mr. W. G. Welchman of Sidney Sussex College for advice about matrices. They have again to thank Mr. J. C. Manisty for valuable assistance at the proof stage.

A. R.

June, 1937 C. V. D.

CONTENTS

VOLUME II

CHAPTER X

FINITE SERIES

Some simple series associated with the binomial theorem for a positive integral index have been discussed in Chapter II, and the use of the calculus in connexion with the summation of series has been illustrated. Further extensions are given here.

The Multinomial Theorem

If n is a positive integer and $(x_1+x_2+\ldots+x_m)^n$ is expanded by direct multiplication, every term of the expansion must be of the form $x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}$ where $n_1+n_2+\ldots+n_m=n$. Hence

$$(x_1+x_2+\ldots+x_m)^n=\sum(\lambda x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}).$$

The value of λ depends on $n_1, n_2, \ldots, n_m$ and is the number of times that the term $x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}$ occurs in the product

$$(x_1+x_2+\ldots+x_m)(x_1+x_2+\ldots+x_m)\ldots(x_1+x_2+\ldots+x_m)$$

where the number of factors is n. But the term occurs once for each possible way of selecting x_1 from n_1 brackets, x_2 from n_2 brackets, x_3 from n_3 brackets, ... , and hence (see p. 7),

$$\lambda=\frac{n!}{n_1!\,n_2!\ldots n_m!}.$$

$$\therefore\ (x_1+x_2+\ldots+x_m)^n=\sum\frac{n!}{n_1!\,n_2!\ldots n_m!}x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}$$

where $n_1, n_2, \ldots, n_m$ assume all positive integral and zero values such that $n_1+n_2+\ldots+n_m=n$.

This is called the *multinomial theorem*; its use is illustrated by Example 1 which follows.

Example 1. Find the coefficient of x^7 in the expansion of

$$(1+3x-2x^3)^{10}.$$

The general term of the expansion is

$$\frac{10!}{p!\,q!\,r!}1^p(3x)^q(-2x^3)^r$$

where $p+q+r=10$.

Terms in x^7 are given by $q+3r=7$, that is by $r=0$, $q=7$, $\therefore p=3$; by $r=1$, $q=4$, $\therefore p=5$; and by $r=2$, $q=1$, $\therefore p=7$.

Therefore the coefficient of x^7 is

$$\frac{10!}{3!\,7!}3^7+\frac{10!}{5!\,4!}3^4(-2)+\frac{10!}{7!\,2!}3(-2)^2$$

and this reduces to $2^4.3^3.5.29$.

Example 2. If n is a positive integer and if

$$(1+x+x^2)^n\equiv\sum_0^{2n}a_rx^r,$$

prove that (i) $a_r=a_{2n-r}$; (ii) $a_0+a_1+\dots+a_{n-1}=\frac{1}{2}(3^n-a_n)$; (iii) $(r+1)a_{r+1}=(n-r)a_r+(2n-r+1)a_{r-1}$, $(0<r<2n)$.

(i) $$\sum_0^{2n}a_r\left(\frac{1}{x}\right)^r\equiv\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n\equiv\frac{1}{x^{2n}}(x^2+x+1)^n.$$

$$\therefore\ \sum_0^{2n}a_rx^{2n-r}\equiv(x^2+x+1)^n\equiv\sum_0^{2n}a_rx^r\equiv\sum_0^{2n}a_{2n-r}x^{2n-r}.$$

$$\therefore\ a_r=a_{2n-r}$$

(ii) Put $x=1$, thus $(1+1+1)^n=a_0+a_1+\dots+a_{2n}$.

But $a_{2n}=a_0$, $a_{2n-1}=a_1$, ..., $a_{n+1}=a_{n-1}$

$$\therefore\ 3^n=2(a_0+a_1+\dots+a_{n-1})+a_n$$

$$\therefore\ a_0+a_1+\dots+a_{n-1}=\tfrac{1}{2}(3^n-a_n).$$

(iii) By differentiation,

$$n(1+x+x^2)^{n-1}(1+2x)\equiv\sum_0^{2n}ra_rx^{r-1},$$

$$\therefore\ n(1+2x)\sum_0^{2n}a_rx^r\equiv(1+x+x^2)\sum_0^{2n}ra_rx^{r-1}.$$

Equating coefficients of x^r $(0<r<2n)$,

$$na_r+2na_{r-1}=(r+1)a_{r+1}+ra_r+(r-1)a_{r-1}$$

$$\therefore\ (r+1)a_{r+1}=(n-r)a_r+(2n-r+1)a_{r-1}.$$

The coefficient of x^r in the expansion of $(1+x)^n$ is usually denoted by ${}_nC_r$ or by $\binom{n}{r}$. For brevity in this chapter it is denoted by c_r when $1 < r < n$; also c_0 stands for 1.

Example 3. Evaluate $\sum\limits_1^n r^3 c_r x^r$

Writing $\sum$ for $\sum\limits_0^n$ and D for $\frac{d}{dx}$, we have $\sum c_r x^r = (1+x)^n$,

$$\therefore \sum r c_r x^r = xD(1+x)^n = nx(1+x)^{n-1}$$

and
$$\sum r^2 c_r x^r = xD\{nx(1+x)^{n-1}\}$$

and
$$\sum r^3 c_r x^r = xD[xD\{nx(1+x)^{n-1}\}].$$

Hence, performing the differentiations,

$$\sum r^3 c_r x^r = nx(1+x)^{n-3}\{1+(3n-1)x+n^2x^2\}.$$

Series of this kind are sometimes called *integro-binomial* series. A more general result is given in Exercise Xa, No. 33.

EXERCISE Xa

Calculus methods are suitable for questions marked *.

A

1. Find the coefficients of a^2bcd and a^3b^2 in $(a+b+c+d)^5$.

2. Expand $(a+b+c)^6$.

3. Find the coefficient of x^5 in the expansion of

$$(1+3x+2x^2)^4.$$

*4. Evaluate $1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+4\left(1+\frac{1}{n}\right)^3+\ldots$ to n terms.

5. Find the value of $\sum 1/(p!\,q!\,r!)$, where the summation extends to all positive integral and zero values of p, q, r such that $p+q+r=n$.

*6. Evaluate $\frac{1}{2}c_0+\frac{1}{3}c_1+\frac{1}{4}c_2+\ldots$ to $n+1$ terms.

*7. If $(1+x+x^2+\ldots+x^m)^n \equiv \sum\limits_0^{mn} a_r x^r$, prove that

$$\sum_1^{mn} r a_r = \tfrac{1}{2}mn(m+1)^n$$

8. Prove that

$$\binom{n}{r} - c_1\binom{n-1}{r-1} + c_2\binom{n-2}{r-2} - \dots \text{ to } r+1 \text{ terms} = 0.$$

*9. Prove that $\sum_{1}^{n+1} (r^2 - 2r + 2)c_{r-1} = (n^2 + n + 4)2^{n-2}$

10. [*Vandermonde's theorem*] Differentiate the product $x^p x^q$ n times by Leibniz' theorem and deduce that if n is any positive integer,

$$[p+q]_n = [p]_n + c_1[p]_{n-1}[q]_1 + c_2[p]_{n-2}[q]_2 + \dots$$

where $[m]_r \equiv m(m-1)(m-2)\dots(m-r+1)$.

11. If $(1 + x + \frac{1}{2}x^2)^n \equiv \sum_{0}^{2n} a_r x^r$, prove that

(i) $a_1 + a_3 + \dots + a_{2n-1} = (5^n - 1)/2^{n+1}$

(ii) $2(r+1)a_{r+1} + 2(r-n)a_r + (r-1-2n)a_{r-1} = 0,\ 1 < r < 2n-1$.

12. If m is a positive integer, prove that

(i) $0 < (3 - \sqrt{7})^m < 1$,

(ii) $(3 + \sqrt{7})^m + (3 - \sqrt{7})^m$ is an even integer,

(iii) if $(3 + \sqrt{7})^m = N + F$, where N is an integer and $0 < F < 1$, then $(3 - \sqrt{7})^m = 2^m/(N + F) < 1$.

Hence deduce that $F + 2^m/(N + F) = 1$ and that N is odd.

B

13. Find the coefficient of $abcde$ in $(a + b + c + d + e + f)^5$.

Find the coefficients of the named powers of x in the expansions of the functions in Nos. 14-16.

14. $(1 + 2x - \frac{1}{2}x^2)^9$; x^5. 15. $(1 + 3x + 2x^3)^7$; x^{10}.

16. $(1 - 2x + 3x^2 - 4x^3)^4$; x^9.

17. Prove that $\sum_{r=0}^{n} \frac{1}{r!\,(n-r)!} = \frac{2^n}{n!}$

*18. Evaluate $\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \dots$ to n terms.

19. Prove that $c_0 - c_1\frac{1+x}{1+nx} + c_2\frac{1+2x}{(1+nx)^2} - c_3\frac{1+3x}{(1+nx)^3} + \dots = 0$.

*20. Evaluate $\frac{1}{3}c_0 - \frac{1}{4}c_1 + \frac{1}{5}c_2 - \dots$ to $n+1$ terms.

21. If $(1 + x + x^2)^{2n} \equiv \sum_{0}^{4n} a_r x^r$, prove that

$$a_0 + a_2 + a_4 + \dots + a_{2n} = \tfrac{1}{4}(9^n + 1 + 2a_{2n}).$$

22. Prove that

$$\frac{1}{m!}+c_1\frac{n}{(m+1)!}+c_2\frac{n(n-1)}{(m+2)!}+\ldots+c_n\frac{n(n-1)\ldots 2.1}{(m+n)!}$$
$$=\frac{(m+n+1)(m+n+2)\ldots(m+2n)}{(m+n)!}$$

[Use $(1+x)^{m+n}(1+x)^n \equiv (1+x)^{m+2n}$]

*23. Sum to $n+1$ terms:

$$\tfrac{1}{2}-\tfrac{1}{3}c_1x+\tfrac{1}{4}c_2x^2-\tfrac{1}{5}c_3x^3+\ldots$$

24. If $(3\sqrt{2}+4)^{2m}=N+F$, where m, N are positive integers and $0<F<1$, prove that $N=(3\sqrt{2}+4)^{2m}+(3\sqrt{2}-4)^{2m}-1$.

C

25. Prove that

$$c_0^2+2c_1^2+3c_2^2+\ldots+(n+1)c_n^2=\frac{(2n-1)!\,(n+2)}{(n-1)!\,n!}$$

26. Prove that

(i) $\binom{n}{r}+c_1\binom{n-1}{r-1}+c_2\binom{n-2}{r-2}+\ldots$ to $r+1$ terms $=2^r c_r$

(ii) $\binom{n}{r}+c_1\binom{n}{r-1}+c_2\binom{n}{r-2}+\ldots$ to $r+1$ terms $=\dfrac{(2n)!}{r!\,(2n-r)!}$

*27. Prove that $\sum\limits_0^n(-1)^r(n-r)(n-r+1)(n-r+2)c_r=0$, if $n>3$.

28. If $(1-x+x^2)^n\equiv\sum\limits_0^{2n}a_rx^r$, prove that

$$a_r+c_1a_{r-1}+c_2a_{r-2}+\ldots \text{ to } r+1 \text{ terms}$$

equals $c_{r/3}$ if r is a multiple of 3, and otherwise is zero.

29. With the notation of No. 10, if n is any positive integer, prove that

$$[a+b+c]_n=\Sigma\frac{n!}{p!\,q!\,r!}[a]_p[b]_q[c]_r,$$

where the summation extends to all positive integral and zero values of p, q, r such that $p+q+r=n$.

30. Find the sum to $r+1$ terms of

$$\binom{3n}{r}-\frac{3n}{1}\binom{3n-2}{r-1}+\frac{3n(3n-3)}{1.2}\binom{3n-4}{r-2}$$
$$-\frac{3n(3n-3)(3n-6)}{1.2.3}\binom{3n-6}{r-3}+\ldots$$

***31.** Prove that the sum to $n+1$ terms of

$$\frac{c_0}{n(n+1)} - \frac{c_1}{(n+1)(n+2)} + \frac{c_2}{(n+2)(n+3)} - \dots$$

equals $\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$, and evaluate the integral.

***32.** Sum to $n+1$ terms

$$1 + c_1\frac{p}{p-n+1} + c_2\frac{p(p-1)}{(p-n+1)(p-n+2)} + \dots \; (p > n-1)$$

***33.** If

$$F_k\{r, y\} \equiv a_0 + a_1 ry + a_2 r(r-1)y^2 + \dots + a_k r(r-1)\dots(r-k+1)y^k,$$

prove that $\sum_{r=0}^{n} F_k\{r, 1\} c_r x^r = (1+x)^n F_k\{n, x/(1+x)\}$.

[Differentiate $(1+x)^n = 1 + c_1 x + c_2 x^2 + \dots + x^n$, k times in succession, multiply respectively by a_0, $a_1 x$, $a_2 x^2$, ..., and add.]

Deduce from this result the values of

$$\text{(i)} \sum_0^n r^3 c_r x^r \qquad \text{(ii)} \sum_0^n (r^4 - r^2) c_r x^r.$$

***34.** Differentiate n times the product $e^x \times e^{ax} \cos bx$ by Leibniz' theorem, and deduce that the sum to $n+1$ terms of

$$\cos\theta + c_1 r\cos(\theta+\phi) + c_2 r^2 \cos(\theta + 2\phi) + \dots$$

equals $\rho^n \cos(\theta + n\lambda)$, where $\rho\sin\lambda = r\sin\phi$ and $\rho\cos\lambda = r\cos\phi + 1$.

Method of Differences. Some examples of the use of difference methods have been given in Chapter III, pp. 37-43. We add further examples here, and shall discuss the method on more general lines later. See pp. 210-220.

The method of finding the sum to n terms of the series

$$u_1 + u_2 + u_3 + \dots$$

when the general term u_r can be expressed in the form

$$f(r+1) - f(r),$$

was explained on pp. 37, 38. Examples 4, 5 below should be compared with Example 2, p. 38 and with the summations (i), (ii) on p. 39.

Example 4. Sum to n terms :

$$\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots$$

The r^{th} term u_r is $\dfrac{1}{(3r-2)(3r+1)(3r+4)}$

But $$\frac{1}{(3r-2)(3r+1)}-\frac{1}{(3r+1)(3r+4)}$$

$$=\frac{6}{(3r-2)(3r+1)(3r+4)}=6u_r$$

$$\therefore\ u_r=\tfrac{1}{6}\left\{\frac{1}{(3r-2)(3r+1)}-\frac{1}{(3r+1)(3r+4)}\right\}.$$

Hence, putting $r=1, 2, 3, \ldots, n$ in succession and adding,

$$u_1+u_2+\ldots+u_n=\tfrac{1}{6}\left\{\frac{1}{1.4}-\frac{1}{(3n+1)(3n+4)}\right\}$$

Note. It follows that the sum to infinity (see p. 57) of this series is $\frac{1}{24}$.

Example 5. Sum to n terms :

$$2.5+5.8+8.11+\ldots$$

The r^{th} term u_r is $(3r-1)(3r+2)$.

But $$(3r-1)(3r+2)(3r+5)-(3r-4)(3r-1)(3r+2)$$

$$=(3r-1)(3r+2)\{(3r+5)-(3r-4)\}=9u_r.$$

$$\therefore\ u_r=\tfrac{1}{9}\{(3r-1)(3r+2)(3r+5)-(3r-4)(3r-1)(3r+2)\}.$$

Hence, putting $r=1, 2, 3, \ldots, n$ in succession and adding,

$$u_1+u_2+\ldots+u_n=\tfrac{1}{9}\{(3n-1)(3n+2)(3n+5)-(-1)(2)(5)\}$$

$$=3n^3+6n^2+n.$$

Examples 4 and 5 illustrate two general types of series to which many others can be reduced. In Example 5, all terms contain the same number of factors which are successive terms of an A.P. and the initial factors are successive terms of the same A.P. In Example 4, the terms are the reciprocals of terms of the type illustrated in Example 5. The methods given above for discovering the required difference can be applied similarly to all series of these types.

The reader should now work Exercise Xb, Nos. 1-4.

The sums of the series $\sum_1^n r(r+1)$, $\sum_1^n r(r+1)(r+2)$, etc. were obtained on p. 39 by the method of Example 5. Any series whose r^{th} term is a rational integral function of r can be summed by reduction to this type (cf. Example 3, p. 40).

Example 6. Find the value of $\sum_1^n (4r^3 - 6r^2 - 4r + 3)$.

If we write $4r^3 - 6r^2 - 4r + 3$ in the form

$$a_0 + a_1 r + a_2 r(r+1) + a_3 r(r+1)(r+2),$$

the required sum is

$$a_0 n + a_1 \sum_1^n r + a_2 \sum_1^n \{r(r+1)\} + a_3 \sum_1^n \{r(r+1)(r+2)\}$$

and by p. 39 the sum is

$$a_0 n + \tfrac{1}{2}a_1 n(n+1) + \tfrac{1}{3}a_2 n(n+1)(n+2) + \tfrac{1}{4}a_3 n(n+1)(n+2)(n+3).$$

The values of a_3, a_2, a_1, a_0 can be written down in that order by inspection, thus

a_3 is the coefficient of r^3, $\therefore\ a_3 = 4$;
$a_2 + 3a_3$ is the coefficient of r^2, that is -6, $\therefore\ a_2 = -18$;
$a_1 + a_2 + 2a_3$ is the coefficient of r, that is -4, $\therefore\ a_1 = 6$;
a_0 is the term independent of r, $\therefore\ a_0 = 3$.

Hence the sum equals

$$3n + 3n(n+1) - 6n(n+1)(n+2) + n(n+1)(n+2)(n+3)$$

which reduces to $n^2(n^2 - 4)$.

Note. The values of a_0, a_1, a_2, a_3 are the remainders obtained by dividing $4r^3 - 6r^2 - 4r + 3$ in succession by r, $r+1$, $r+2$, and may therefore be found by the method given on p. 161, if not easily determined by inspection. The working would then be as follows:

$$\begin{array}{llll} 4 & -6 & -4 & +3 \\ 4 & -10 & +6 & \\ 4 & -18 & & \end{array}$$

$$\therefore\ a_0 = 3,\ a_1 = 6,\ a_2 = -18,\ a_3 = 4.$$

The series in Example 6 can also be summed by using the values of $\sum r$, $\sum r^2$, $\sum r^3$. See pp. 38, 39. Thus

$$\sum_1^n (4r^3 - 6r^2 - 4r + 3) = 4\sum r^3 - 6\sum r^2 - 4\sum r + 3n$$
$$= n^2(n+1)^2 - n(n+1)(2n+1) - 2n(n+1) + 3n$$
$$= n^2(n^2 - 4).$$

This method is however inconvenient for functions of higher degree.

The reader should now work Exercise Xb, Nos. 5-7.

Examples 7-9 illustrate other series which can be reduced to the type summed in Example 4.

Example 7. Sum to n terms:

$$\frac{2}{3.7.11} + \frac{5}{7.11.15} + \frac{8}{11.15.19} + \ldots$$

The r^{th} term $= \dfrac{3r-1}{(4r-1)(4r+3)(4r+7)} = \dfrac{\frac{3}{4}(4r-1) - \frac{1}{4}}{(4r-1)(4r+3)(4r+7)}$

$$= \frac{\frac{3}{4}}{(4r+3)(4r+7)} - \frac{\frac{1}{4}}{(4r-1)(4r+3)(4r+7)}$$

The two series whose r^{th} terms are

$$\frac{1}{(4r+3)(4r+7)} \quad \text{and} \quad \frac{1}{(4r-1)(4r+3)(4r+7)}$$

can now be summed by the method of Example 4, and the reader should verify that the sums are

$$\frac{1}{4}\left(\frac{1}{7} - \frac{1}{4n+7}\right) \quad \text{and} \quad \frac{1}{8}\left\{\frac{1}{3.7} - \frac{1}{(4n+3)(4n+7)}\right\}.$$

Hence the sum to n terms of the given series is

$$\frac{3}{4}\cdot\frac{1}{4}\left(\frac{1}{7} - \frac{1}{4n+7}\right) - \frac{1}{4}\cdot\frac{1}{8}\left\{\frac{1}{3.7} - \frac{1}{(4n+3)(4n+7)}\right\}$$

which reduces to $\dfrac{17}{672} - \dfrac{24n+17}{32(4n+3)(4n+7)}$.

Note. The sum to infinity of the series is $\dfrac{17}{672}$.

Example 8. Sum to n terms:

$$\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\ldots$$

The r^{th} term u_r is $\dfrac{r+2}{r(r+1)(r+3)}$. Multiply the numerator and denominator by $r+2$ to make the factors of the denominator successive terms of an A.P.,

$$\therefore\ u_r=\frac{r^2+4r+4}{r(r+1)(r+2)(r+3)}=\frac{r(r+1)+3r+4}{r(r+1)(r+2)(r+3)}$$

$$=\frac{1}{(r+2)(r+3)}+\frac{3}{(r+1)(r+2)(r+3)}+\frac{4}{r(r+1)(r+2)(r+3)}.$$

$\therefore$ continuing as in Example 7, we obtain

$$\sum_1^n u_r=\left(\frac{1}{3}-\frac{1}{n+3}\right)+\frac{3}{2}\left\{\frac{1}{2.3}-\frac{1}{(n+2)(n+3)}\right\}$$
$$+\frac{4}{3}\left\{\frac{1}{1.2.3}-\frac{1}{(n+1)(n+2)(n+3)}\right\}$$

which reduces to $\dfrac{29}{36}-\dfrac{6n^2+27n+29}{6(n+1)(n+2)(n+3)}$.

Alternatively the series can be summed by expressing the general term in partial fractions as in Example 9, with which Example 5 (p. 116) should be compared.

Example 9. Sum to n terms:

$$\frac{1}{2.4.6}+\frac{2}{3.5.7}+\frac{3}{4.6.8}+\ldots$$

$$u_r=\frac{r}{(r+1)(r+3)(r+5)}=\frac{1}{8}\left(-\frac{1}{r+1}+\frac{6}{r+3}-\frac{5}{r+5}\right).$$

$$\therefore\ 8\sum_1^n u_r=-\left(\frac{1}{2}+\ldots+\frac{1}{n+1}\right)+6\left(\frac{1}{4}+\ldots+\frac{1}{n+3}\right)$$
$$-5\left(\frac{1}{6}+\ldots+\frac{1}{n+5}\right)$$

$$=-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)+6\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{n+2}+\frac{1}{n+3}\right)$$
$$-5\left(\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\frac{1}{n+5}\right)$$

Hence $\sum_1^n u_r = \frac{17}{96} + \frac{1}{8}\left(\frac{1}{n+2} + \frac{1}{n+3} - \frac{5}{n+4} - \frac{5}{n+5}\right)$

The reader should now work Exercise Xb, Nos. 8-10.

The general terms of the series summed in Examples 4-9 can each be reduced to one of the following forms :

(i) $u_r = \{a+(r-1)d\}\{a+rd\}\ldots\{a+(r+k-2)d\}$

(ii) $u_r = \dfrac{1}{\{a+(r-1)d\}\{a+rd\}\ldots\{a+(r+k-2)d\}}$, $k>1$

where each term contains k factors which are successive terms of an A.P. and the initial factors are successive terms of the same A.P.

For type (i), see Example 5 (p. 201). The required difference is obtained by writing down u_r and inserting the next factor at the end.

If $v_r = \{a+(r-1)d\}\{a+rd\}\ldots\{a+(r+k-2)d\}\{a+(r+k-1)d\}$,

$$v_r - v_{r-1} = u_r[\{a+(r+k-1)d\} - \{a+(r-2)d\}] = u_r(k+1)d$$

$$\therefore\ u_1+u_2+\ldots+u_n = (v_n - v_0)/\{(k+1)d\}.$$

The sum can therefore be obtained by the following rule :

Write down the n^{th} term and insert the next factor at the end; divide the result by the increased number of factors and by the common difference. Then subtract from it the expression obtained by putting 0 for n.

For type (ii), see Example 4 (p. 201). The required difference is obtained by writing down u_r and removing the first factor.

If $v_r = \dfrac{1}{\{a+rd\}\ldots\{a+(r+k-2)d\}}$, $k>1$,

$$v_r - v_{r-1} = u_r[\{a+(r-1)d\} - \{a+(r+k-2)d\}] = -u_r(k-1)d;$$

$$\therefore\ u_1+u_2+\ldots+u_n = -(v_n - v_0)/\{(k-1)d\}.$$

The sum can therefore be obtained by the following rule :

Write down the n^{th} term and remove the first factor; divide the result by the diminished number of factors and by the common difference and change the sign. Then subtract from it the expression obtained by putting 0 for n.

Thus in Example 7 (p. 203) the n^{th} term can be expressed in the form

$$\frac{\frac{3}{4}}{(4n+3)(4n+7)}-\frac{\frac{1}{4}}{(4n-1)(4n+3)(4n+7)}.$$

$\therefore$ the sum to n terms is $f(n)-f(0)$, where

$$f(n)=-\frac{1}{1.4}\cdot\frac{\frac{3}{4}}{4n+7}+\frac{1}{2.4}\cdot\frac{\frac{1}{4}}{(4n+3)(4n+7)}=-\frac{24n+17}{32(4n+3)(4n+7)}.$$

$\therefore$ the sum to n terms is $\dfrac{17}{32.3.7}-\dfrac{24n+17}{32(4n+3)(4n+7)}$.

No use should be made of these rules until the methods for obtaining the sums from first principles are fully understood and have been practised thoroughly. Many prefer to disregard such rules altogether.

Another type of series is illustrated in Examples 10, 11.

Example 10. Sum to n terms :

$$\frac{4}{5}+\frac{4.7}{5.8}+\frac{4.7.10}{5.8.11}+\ldots$$

The r^{th} term u_r is $\dfrac{4.7.10\ldots(3r+1)}{5.8.11\ldots(3r+2)}$.

But for $r \geqslant 2$, $\dfrac{4.7\ldots(3r+1)(3r+4)}{5.8\ldots(3r+2)}-\dfrac{4.7\ldots(3r-2)(3r+1)}{5.8\ldots(3r-1)}$

$$=\frac{4.7\ldots(3r+1)}{5.8\ldots(3r+2)}\{(3r+4)-(3r+2)\}=2u_r.$$

$$\therefore\ u_r=\tfrac{1}{2}\left\{\frac{4.7\ldots(3r+1)(3r+4)}{5.8\ldots(3r+2)}-\frac{4.7\ldots(3r-2)(3r+1)}{5.8\ldots(3r-1)}\right\}.$$

Hence, putting $r=2, 3, \ldots, n$ in succession and adding

$$\sum_1^n u_r=u_1+\tfrac{1}{2}\left\{\frac{4.7\ldots(3n+1)(3n+4)}{5.8\ldots(3n+2)}-\frac{4.7}{5}\right\}$$

$$=\frac{4.7\ldots(3n+1)(3n+4)}{2.5\ldots(3n-1)(3n+2)}-2.$$

The tests given in Chapter IV are not sufficiently refined to determine whether this series is convergent or divergent. A proof that this series is in fact divergent is given on p. 350 in Example 16.

The method used in Example 10 would not have been successful if the common differences in the numerator and denominator had not been the same. The series in Example 11 is substantially of the same type. A more general form is given in Exercise Xb, No. 40.

Example 11. Sum to n terms :

$$\frac{p}{2}-\frac{p(p-1)}{2.5}3+\frac{p(p-1)(p-2)}{2.5.8}3^2-\dots$$

Since this series can be written in the form

$$-\tfrac{1}{3}\left\{\frac{(-3p)}{2}+\frac{(-3p)(-3p+3)}{2.5}+\frac{(-3p)(-3p+3)(-3p+6)}{2.5.8}+\dots\right\}$$

it is of the same type as the series in Example 10 and can be summed in the same way. If for simplicity a is written for $-3p$, the r^{th} term of the series in the brackets can be written

$$\frac{1}{a+1}\left\{\frac{a(a+3)\dots(a+3r-3)(a+3r)}{2.5\dots(3r-1)}-\frac{a(a+3)\dots(a+3r-6)(a+3r-3)}{2.5\dots(3r-4)}\right\},\quad r>2,$$

and it then follows that the sum to n terms of the given series is

$$\left\{(-3)^n\frac{p(p-1)\dots(p-n+1)(p-n)}{2.5\dots(3n-1)}-p\right\}\Big/(1-3p).$$

The reader should verify these statements.

EXERCISE Xb

A

Find the sum to n terms and the sum to infinity when it exists of the series in Nos. 1-10.

1. $\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\dots$

2. $\frac{1}{8.11.14}+\frac{1}{11.14.17}+\frac{1}{14.17.20}+\ldots$

3. $1.4+4.7+7.10+\ldots$

4. $2.7.12+7.12.17+12.17.22+\ldots$

5. $1.4+2.7+3.10+\ldots$

6. $1.3.5+2.4.6+3.5.7+\ldots$

7. r^{th} term, $r(r+1)(2r+1)$

8. r^{th} term, $\frac{2r-1}{r(r+1)(r+2)}$

9. r^{th} term, $\frac{1}{r(r+2)}$

10. r^{th} term, $\frac{r+1}{r(r+2)(r+3)}$

11. Sum to n terms : $\frac{3}{8}+\frac{3.5}{8.10}+\frac{3.5.7}{8.10.12}+\ldots$

12. Sum to n terms : $\frac{1}{2.4}+\frac{1.3}{2.4.6}+\frac{1.3.5}{2.4.6.8}+\ldots$

B

Find the sum to n terms and the sum to infinity when it exists of the series in Nos. 13-22.

13. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots$

14. $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots$

15. $5.7+7.9+9.11+\ldots$

16. $1.5.9+5.9.13+9.13.17+\ldots$

17. $2.4+5.7+8.10+11.13+\ldots$

18. $1.1+4.5+7.9+10.13+\ldots$

19. r^{th} term, $\frac{r}{(2r-1)(2r+1)(2r+3)}$

20. r^{th} term, $\frac{1}{(2r-1)(2r+3)}$

21. r^{th} term, $\frac{3r+1}{(r+1)(r+2)(r+3)}$

22. r^{th} term, $\frac{1}{r(r+1)(r+3)}$

23. Sum to n terms : $\frac{2}{3}+\frac{2.5}{3.6}+\frac{2.5.8}{3.6.9}+\ldots$

24. Sum to n terms : $\frac{3.8}{4.9}+\frac{3.8.13}{4.9.14}+\frac{3.8.13.18}{4.9.14.19}+\ldots$

C

Sum to n terms the series whose r^{th} terms are as follows:

25. $\dfrac{1}{r(r+1)(r+2)(r+3)}$

26. $\dfrac{1}{\{a+(r-1)d\}\{a+rd\}\{a+(r+1)d\}}$, $d \neq 0$

27. $r(r+1)(r+2)(r+3)$ 28. $\{a+(r-1)d\}\{a+rd\}$

29. $r^2(4r-1)$ 30. r^4 31. $r(r+2)(r+5)$

32. $r^2(r^2-1)$ 33. $r^2(r+1)(r+2)$

34. $\dfrac{(r+1)^2}{r(r+2)}$ 35. $\dfrac{1}{r(r+2)(r+4)(r+5)}$ 36. $\dfrac{r+1}{r(r+2)(r+3)(r+4)}$

Sum to n terms the series in Nos. 37-39.

37. $\dfrac{1}{p}+\dfrac{2!}{p(p+1)}+\dfrac{3!}{p(p+1)(p+2)}+\dots$

38. $\dfrac{1}{p-1}-\dfrac{2!}{(p-1)(p-2)}+\dfrac{3!}{(p-1)(p-2)(p-3)}-\dots$

39. $\dfrac{p}{1}2-\dfrac{p(p-1)}{1.3}2^2+\dfrac{p(p-1)(p-2)}{1.3.5}2^3-\dots$

40. Write down the sum to n terms of

$$(a_1-1)+a_1(a_2-1)+a_1a_2(a_3-1)+a_1a_2a_3(a_4-1)+\dots$$

and by substituting $\dfrac{a}{b}$ for a_1, $\dfrac{a+d_1}{b+d_1}$ for a_2, ... $\dfrac{a+d_{r-1}}{b+d_{r-1}}$ for a_r, find the sum to n terms of

$$\frac{a}{b+d_1}+\frac{a(a+d_1)}{(b+d_1)(b+d_2)}+\frac{a(a+d_1)(a+d_2)}{(b+d_1)(b+d_2)(b+d_3)}+\dots$$

Deduce from the result the sums of the series in Nos. 11, 12.

41. Deduce from No. 40 the sum to n terms of

(i) $\dfrac{a}{b}+\dfrac{a(a+d)}{b(b+d)}+\dfrac{a(a+d)(a+2d)}{b(b+d)(b+2d)}+\dots$

(ii) $\dfrac{a}{b}-\dfrac{a(a-d)}{b(b+d)}+\dfrac{a(a-d)(a-2d)}{b(b+d)(b+2d)}-\dots$

Difference Notation. We here discuss the subject on more general lines than hitherto, and introduce the "difference notation".

From any series

$$u_1, u_2, u_3, \dots, u_r, \dots$$

may be constructed a series of differences

$$u_2 - u_1, u_3 - u_2, u_4 - u_3, \dots, u_{r+1} - u_r, \dots$$

We write

$$u_2 - u_1 = \Delta u_1,\ u_3 - u_2 = \Delta u_2, \dots, u_{r+1} - u_r = \Delta u_r, \dots$$

and call the series

$$\Delta u_1, \Delta u_2, \Delta u_3, \dots, \Delta u_r, \dots$$

the series of *first differences.*

Similarly, writing $\Delta^2 u_r$ for $\Delta u_{r+1} - \Delta u_r$, we call

$$\Delta^2 u_1, \Delta^2 u_2, \Delta^2 u_3, \dots, \Delta^2 u_r, \dots$$

the series of *second differences,* and in a similar way we can form series of 3rd, 4th, ..., kth, ... differences.

The series of first differences formed from the series of kth differences of the original series is the series of $(k+1)$th differences of the original series, that is

$$\Delta(\Delta^k u_r) = \Delta^{k+1} u_r$$

and more generally

$$\Delta^l(\Delta^k u_r) = \Delta^{k+l} u_r$$

It is sometimes convenient to use

$$\Delta^{-1} u_1,\ \Delta^{-1} u_2,\ \Delta^{-1} u_3, \dots$$

for the series which has

$$u_1, u_2, u_3, \dots$$

as its series of first differences, and then

$$\begin{aligned} u_1 &= \Delta^{-1} u_2 - \Delta^{-1} u_1 \\ u_2 &= \Delta^{-1} u_3 - \Delta^{-1} u_2 \\ &\vdots \\ u_n &= \Delta^{-1} u_{n+1} - \Delta^{-1} u_n \end{aligned}$$

$\therefore$ by addition $\quad u_1 + u_2 + \dots + u_n = \Delta^{-1} u_{n+1} - \Delta^{-1} u_1.$

Thus the problem of finding the sum to n terms of a series $u_1 + u_2 + u_3 + \ldots$ is equivalent to the problem of finding $\Delta^{-1}u_{n+1}$ or $\Delta^{-1}u_n$.

We have already in effect used this method of summation in many examples; e.g. in Example 5 (p. 201) $u_n = (3n-1)(3n+2)$, and the argument amounted to showing that

$$\Delta^{-1}u_{n+1} = \tfrac{1}{9}(3n-1)(3n+2)(3n+5).$$

The process of finding $\Delta^{-1}u_n$ may be compared with that of integration, and just as there are many elementary functions $f(x)$ of a continuous variable x such that $\int f(x)dx$ cannot be expressed in terms of elementary functions, so there are many elementary functions $f(n)$ of a positive integral variable n such that $\Delta^{-1}f(n)$ cannot be expressed as an elementary function of n.

We shall consider in this chapter some general types of function $f(n)$ for which $\Delta^{-1}f(n)$ can easily be found.

$\Delta^{-1}f(n)$ contains an arbitrary constant, because if

$$\phi(n+1) - \phi(n) \equiv u_n,$$

then also $$\{\phi(n+1)+c\} - \{\phi(n)+c\} \equiv u_n.$$

EXERCISE Xc

A

Find the r^{th} term of the first difference series of the series in Nos. 1-4.

1. 1.2.3, 2.3.4, 3.4.5, ...

2. 1!, 2!, 3!, ...

3. $\frac{1}{1.2.3}, \frac{1}{2.3.4}, \frac{1}{3.4.5}, \ldots$

4. $\sin\theta, \sin 3\theta, \sin 5\theta, \ldots$

Find the r^{th} term of the second difference series of the series in Nos. 5, 6.

5. $1^3, 2^3, 3^3, \ldots$

6. $a, ax, ax^2, \ldots$

Find $\Delta^{-1}u_n$ and $\Delta^{-1}u_{n+1} - \Delta^{-1}u_1$, when u_r has the values given in Nos. 7-11, (see Examples 4-10, pp. 201-206):

7. r

8. $r(r+1)$

9. x^r

10. $\dfrac{1}{(2r+1)(2r+3)}$

11. $\dfrac{4.7.10\ldots(3r+1)}{5.8.11\ldots(3r+2)}$

12. Evaluate $\sum_1^n (r^2+1)(r!)$

13. Evaluate $\sum_1^n \dfrac{r}{(r+1)(r+2)} 2^r$

B

Find the r^{th} term of the first difference series of the series in Nos. 14, 15.

14. $1^2, 2^2, 3^2, \ldots$

15. $\dfrac{1}{2!}, \dfrac{1}{3!}, \dfrac{1}{4!}, \ldots$

Find the r^{th} term of the second difference series of the series in Nos. 16, 17.

16. $1.2.3,\ 2.3.4,\ 3.4.5, \ldots$

17. $\log 1, \log 2, \log 3, \ldots$

Find $\Delta^{-1}u_n$ and $\Delta^{-1}u_{n+1} - \Delta^{-1}u_1$, when u_r has the values given in Nos. 18-20.

18. $(4r-3)(4r+1)$

19. $\dfrac{1}{r(r+1)}$

20. $\log\left(1+\dfrac{1}{r}\right)$

21. Evaluate $\sum_1^n \dfrac{r}{(r+2)!} 2^r$

22. Evaluate $\sum_1^n \dfrac{r+3}{r(r+1)} \left(\dfrac{2}{3}\right)^r$

C

23. If $u_r = \sin\theta \sin r\theta$, find $\Delta^{-1}u_n$ and $\Delta^{-1}u_{n+1} - \Delta^{-1}u_1$.

Find the sum to n terms and the sum to infinity of the series whose r^{th} terms are the functions in Nos. 24-26.

24. $\dfrac{r^2+r-1}{(r+2)!}$

25. $\dfrac{2x+2r-1}{(x+r)^2(x+r-1)^2}$

26. $\dfrac{r+1}{(r+2)!}$

27. Sum to $n+1$ terms :

$$1+\frac{a}{d_1}+\frac{a(a+d_1)}{d_1 d_2}+\frac{a(a+d_1)(a+d_2)}{d_1 d_2 d_3}+\ldots$$

28. Prove that the sum to n terms of

$$\frac{a_1}{a_1+d_1}+\frac{d_1 a_2}{(a_1+d_1)(a_2+d_2)}+\frac{d_1 d_2 a_3}{(a_1+d_1)(a_2+d_2)(a_3+d_3)}+\ldots$$

equals $1 - d_1 d_2 \ldots d_n/\{(a_1+d_1)(a_2+d_2)\ldots(a_n+d_n)\}$

29. Sum to n terms :

$$\frac{2}{1.3}+\frac{4}{1.3.5}+\frac{6}{1.3.5.7}+\frac{8}{1.3.5.7.9}+\ldots$$

Successive Differences

If $u_r = ar^k$ where a and k are constants (independent of r),

$$\Delta u_r = a\{(r+1)^k - r^k\}$$

$$= a\left\{\binom{k}{1} r^{k-1} + \binom{k}{2} r^{k-2} + \ldots + 1\right\}$$

which is a polynomial of degree $k-1$ in r.

Also if $u_r = v_r + w_r$,

it follows from the definition on p. 210 that

$$\Delta u_r = \Delta v_r + \Delta w_r.$$

Hence if u_r is a polynomial of degree k in r,

Δu_r is a polynomial of degree $k-1$ in r,

$\Delta^2 u_r$ is a polynomial of degree $k-2$ in r,

and so on. Hence

$\Delta^{k-1} u_r$ is of the form $ar+b$ where a, b are constants

and $\Delta^k u_r$ is a constant.

Also $\Delta^m u_r = 0$ if $m > k$.

Consider for example the series whose r^{th} term is the cubic polynomial $r^3 + r$. The series is

2, 10, 30, 68, 130, 222, 350, ...

and the successive difference series are

8, 20, 38, 62, 92, 128, ...

12, 18, 24, 30, 36, ...

6, 6, 6, 6, ...

0, 0, 0, ...

Thus $\Delta^3 u_r = 6$, $\Delta^4 u_r = 0$, for all values of r.

Conversely if $\Delta^4 u_r = 0$ for all values of r, it can be shown that $\Delta^3 u_r$ is constant, $\Delta^2 u_r$ is of the first degree in r, Δu_r is of the second degree in r, and u_r is a cubic in r.

For example if $\Delta^2 u_r = ar + b$ where a, b are constants (independent of r), then by definition

$$\Delta u_{r+1} - \Delta u_r = \Delta^2 u_r = ar + b$$
$$\therefore \Delta u_r - \Delta u_{r-1} = a(r-1) + b$$
$$\Delta u_{r-1} - \Delta u_{r-2} = a(r-2) + b$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\Delta u_2 - \Delta u_1 = a + b.$$

$\therefore$ by addition, $\Delta u_r - \Delta u_1 = a\{1 + 2 + 3 + \dots + (r-1)\} + (r-1)b$

$$= \tfrac{1}{2}r(r-1)a + (r-1)b.$$

$$\therefore \Delta u_r = cr^2 + dr + e$$

where c, d, e are constants.

The general result is proved on p. 216.

Example 12. Find by the method of differences the r^{th} term of a series which has 10, 11, 14, 21, 34, for its first five terms.

The series and its successive difference series are

10	11	14	21	34
1	3	7	13	
2	4	6		
2	2			

Thus $\Delta^3 u_1 = 2$ and $\Delta^3 u_2 = 2$, and if we assume that $\Delta^3 u_r = 2$ for *all* values of r, the previous discussion suggests that u_r is a cubic in r. That this must in fact be so is proved later (see p. 216). We then have

$$u_r = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$

and it is easy to find a_0, a_1, a_2, a_3, such that u_1, u_2, u_3, u_4 have the values 10, 11, 14, 21 respectively, but it is simpler to write

$$u_r = b_0 + b_1(r-1) + b_2(r-1)(r-2) + b_3(r-1)(r-2)(r-3).$$

Putting $r = 1$, $10 = u_1 = b_0$

$r = 2$, $11 = u_2 = b_0 + b_1$, $\therefore b_1 = 1.$

$r = 3$, $14 = u_3 = b_0 + 2b_1 + 2b_2$, $\therefore b_2 = 1.$

$r = 4$, $21 = u_4 = b_0 + 3b_1 + 6b_2 + 6b_3$, $\therefore b_3 = \tfrac{1}{3}$

$$\therefore u_r = 10 + (r-1) + (r-1)(r-2) + \tfrac{1}{3}(r-1)(r-2)(r-3)$$
$$= \tfrac{1}{3}(r^3 - 3r^2 + 5r + 27).$$

The general statement made above may be checked by noting that for this form of u_r

$$u_5 = \tfrac{1}{3}(125 - 75 + 25 + 27) = 34.$$

The assumption that $\Delta^3 u_r = 2$ for all values of r fixes the values of all the other terms of the series which begins 10, 11, 14, 21, 34. But if we are merely given these first five terms and do not make any assumption, we can continue the series in an arbitrary manner. For example the series determined by

$$u_r = \tfrac{1}{3}(r^3 - 3r^2 + 5r + 27) + \lambda(r-1)(r-2)(r-3)(r-4)(r-5)$$

for any value of λ, not necessarily a constant, has also 10, 11, 14, 21, 34 for its first five terms.

Example 13. Find by the method of differences the r^{th} term of a series which has 10, 11, 14, 21 for its first four terms.

The series and its successive difference series are

$$\begin{array}{llll} 10 & 11 & 14 & 21 \ldots\ldots \\ 1 & 3 & 7 \ldots\ldots\ldots\ldots & \\ 2 & 4 \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}$$

Here $\Delta^2 u_1 = 2$, $\Delta^2 u_2 = 4$ and if we assume $\Delta^2 u_r = 2r$, we obtain the series in Example 12 for which

$$u_r = \tfrac{1}{3}(r^3 - 3r^2 + 5r + 27).$$

Alternatively we might assume that $\Delta^2 u_r = 2^r$ and we then have as on p. 214 since $\Delta u_{r+1} - \Delta u_r = \Delta^2 u_r = 2^r$,

$$\Delta u_r - \Delta u_{r-1} = 2^{r-1},\ \Delta u_{r-1} - \Delta u_{r-2} = 2^{r-2},\ \ldots,\ \Delta u_2 - \Delta u_1 = 2,$$

$$\therefore\ \Delta u_r - \Delta u_1 = 2^{r-1} + 2^{r-2} + \ldots + 2^2 + 2 = 2^r - 2.$$

But $\Delta u_1 = 1$, $\quad \therefore\ \Delta u_r = 2^r - 1.$

Hence, by the same argument,

$$u_r - u_1 = (2^{r-1} - 1) + (2^{r-2} - 1) + \ldots + (2^2 - 1) + (2 - 1)$$
$$= 2^r - 2 - (r - 1)$$

But $u_1 = 10$, $\quad \therefore\ u_r = 2^r - r + 9$,

which is a solution of quite another form.

The reader should now work Exercise Xd, Nos. 1-5.

General Formulae

It has been pointed out that

$$\text{(i)} \quad \Delta(v_r + w_r) = \Delta v_r + \Delta w_r \qquad \text{(p. 213)}$$

$$\text{(ii)} \quad \Delta^l(\Delta^k u_r) = \Delta^{k+l} u_r \qquad \text{(p. 210)}$$

and these relations show that the symbol Δ operates according to the distributive and index laws of algebra.

Now by definition

$$u_2 = u_1 + \Delta u_1 \quad \text{and} \quad u_3 = u_2 + \Delta u_2$$

hence $u_3 = (u_1 + \Delta u_1) + \Delta(u_1 + \Delta u_1) = (u_1 + \Delta u_1) + (\Delta u_1 + \Delta^2 u_1)$
$= u_1 + 2\Delta u_1 + \Delta^2 u_1.$

But since the symbol Δ obeys the laws of algebra, this work might have been written

$$u_3 = (1+\Delta)u_2 = (1+\Delta)(1+\Delta)u_1$$
$$= (1+\Delta)^2 u_1 = (1 + 2\Delta + \Delta^2)u_1$$
$$= u_1 + 2\Delta u_1 + \Delta^2 u_1.$$

Also $u_{n+1} = (1+\Delta)^n u_1$

i.e. $$\mathbf{u_{n+1} = u_1 + \binom{n}{1} \Delta u_1 + \binom{n}{2} \Delta^2 u_1 + \ldots + \Delta^n u_1}$$

where the coefficients are binomial coefficients.

This is known as *Newton's Difference Formula*, and it is easily remembered in the form $(1+\Delta)^n u_1$.

It follows from this result that if the terms of the kth difference series are all equal (and not zero), u_{n+1} and therefore also u_n is a polynomial of degree k in n.

For if $\Delta^k u_r$ is constant, i.e. independent of r,

then $$\Delta^{k+1} u_r = 0 = \Delta^{k+2} u_r = \ldots .$$

Hence

$$u_{n+1} = u_1 + a_1 n + a_2 n(n-1) + \ldots + a_k n(n-1)(n-2) \ldots (n-k+1),$$

where $a_p = \Delta^p u_1 / p!$

This is the general statement to which reference was made on p. 214.

Sum to n Terms

By applying Newton's Formula to the series

$$\Delta^{-1}u_1,\ \Delta^{-1}u_2,\ \Delta^{-1}u_3, \dots$$

for which the successive difference series are

$$\begin{array}{ccc} u_1, & u_2, & u_3, \dots \\ \Delta u_1, & \Delta u_2, & \Delta u_3, \dots \end{array}$$

we have

$$\Delta^{-1}u_{n+1} = \Delta^{-1}u_1 + \binom{n}{1}u_1 + \binom{n}{2}\Delta u_1 + \dots + \Delta^{n-1}u_1.$$

But $u_r = \Delta^{-1}u_{r+1} - \Delta^{-1}u_r$, $\quad\therefore\ \sum_1^n u_r = \Delta^{-1}u_{n+1} - \Delta^{-1}u_1$,

$$\therefore\ \sum_1^n u_r = \binom{n}{1}u_1 + \binom{n}{2}\Delta u_1 + \dots + \Delta^{n-1}u_1$$

which may be expressed symbolically in the form

$$\sum_1^n u_r = \Delta^{-1}\{(1+\Delta)^n - 1\}u_1.$$

If u_r is a polynomial of degree k in r, we have shown that $\Delta^k u_r$ is a constant and that $\Delta^{k+1}u_r = 0 = \Delta^{k+2}u_r = \dots$. The above relation is then a formula for the sum to n terms of the series. But in other cases it merely transforms one 'sum to n terms' into another. For example with the series $1, x, x^2, \dots, x^{n-1}$, it is easy to see that $\Delta u_1 = x - 1$, $\Delta^2 u_1 = (x-1)^2$, $\Delta^3 u_1 = (x-1)^3, \dots$ and hence

$$1 + x + x^2 + \dots + x^{n-1} = n + \binom{n}{2}(x-1) + \binom{n}{3}(x-1)^2 + \dots + (x-1)^{n-1}.$$

In the above illustration, the values of $\Delta u_1, \Delta^2 u_1, \Delta^3 u_1, \dots$ are obtained quickly by successive calculations. It is however sometimes convenient to have a formula for $\Delta^m u_1$ in terms of $u_1, u_2, \dots, u_{m+1}$.

Since $\quad \Delta u_1 = u_2 - u_1 \quad$ and $\quad \Delta u_2 = u_3 - u_2$,

$$\Delta^2 u_1 = \Delta u_2 - \Delta u_1 = (u_3 - u_2) - (u_2 - u_1),$$

i.e. $\quad \Delta^2 u_1 = u_3 - 2u_2 + u_1.$

Similarly

$$\begin{aligned}\Delta^3 u_1 &= (u_4 - 2u_3 + u_2) - (u_3 - 2u_2 + u_1) \\ &= u_4 - 3u_3 + 3u_2 - u_1\end{aligned}$$

and it is easy to see by induction that

$$\Delta^m u_1 = u_{m+1} - \binom{m}{1} u_m + \binom{m}{2} u_{m-1} - \dots + (-1)^m u_1$$

where the coefficients are binomial coefficients.

A formal proof may also be obtained by applying Newton's Difference Formula to the series

$$u_1, \quad -\Delta u_1, \quad \Delta^2 u_1, \quad -\Delta^3 u_1, \dots$$

which has for successive difference series

$$-u_2, \quad \Delta u_2, \quad -\Delta^2 u_2, \quad \Delta^3 u_2, \dots$$
$$u_3, \quad -\Delta u_3, \quad \Delta^2 u_3, \quad -\Delta^3 u_3, \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

so that the formula gives

$$(-1)^m \Delta^m u_1 = u_1 - \binom{m}{1} u_2 + \binom{m}{2} u_3 - \dots + (-1)^m u_{m+1}$$

$$\therefore \Delta^m u_1 = u_{m+1} - \binom{m}{1} u_m + \binom{m}{2} u_{m-1} - \dots + (-1)^m u_1.$$

Since a series may be formed by starting at any term of a given series, $\Delta^m u_r$ is given by

$$\Delta^m u_r = u_{m+r} - \binom{m}{1} u_{m+r-1} + \binom{m}{2} u_{m+r-2} \dots + (-1)^m u_r$$

and in particular

$$\Delta^2 u_r = u_{r+2} - 2u_{r+1} + u_r$$
$$\Delta^3 u_r = u_{r+3} - 3u_{r+2} + 3u_{r+1} - u_r$$

and so on. Use is made of these results in the solution of certain difference equations; see Examples 4, 6 on pp. 230, 231.

Example 14. Given that the r^{th} term of the series

$$3, \ 4, \ 7, \ 14, \ 27, \ 48, \ 79, \dots$$

is a polynomial in r, find its possible values.

The series and its successive difference series are

3	4	7	14	27	48	79 ...
1	3	7	13	21	31	
2	4	6	8	10		
2	2	2	2			

Hence a possible value of u_r is obtained by assuming that $\Delta^3 u_r = 2$ for all values of r. We then have

$$u_r = (1+\Delta)^{r-1} u_1$$
$$= 3 + (r-1)1 + \frac{(r-1)(r-2)}{1.2}2 + \frac{(r-1)(r-2)(r-3)}{1.2.3}2$$
$$= \tfrac{1}{3}(r^3 - 3r^2 + 5r + 6).$$

Hence by hypothesis the value which u_r can take when the assumption about $\Delta^3 u_r$ is not made, differs from $\frac{1}{3}(r^3 - 3r^2 + 5r + 6)$ by a polynomial in r which is zero for the values $r = 1, 2, \ldots, 6, 7$.

Therefore all possible values of u_r are of the form

$$\tfrac{1}{3}(r^3 - 3r^2 + 5r + 6) + (r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)P(r)$$

where $P(r)$ denotes a polynomial in r.

Example 15. Evaluate $1^3 + 2^3 + 3^3 + \ldots + n^3$.

The series is	1	8	27	64	125 ...
and the difference series are	7	19	37	61	
	12	18	24		
	6	6			

Hence $\Delta u_1 = 7,\quad \Delta^2 u_1 = 12,\quad \Delta^3 u_1 = 6,$

and it is known that $\Delta^4 u_1 = 0 = \Delta^5 u_1 = \ldots$

$$\therefore \sum_1^n r^3 = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$
$$= n + \tfrac{7}{2}n(n-1) + 2n(n-1)(n-2) + \tfrac{1}{4}n(n-1)(n-2)(n-3)$$

which reduces to $\frac{1}{4}n^2(n+1)^2$.

Note. Other methods for summing this series are given on pp. 39, 40, 42.

The reader should now work Exercise Xd, Nos. 6-8.

For the G.P. $a, ax, ax^2, ax^3, \ldots$ of common ratio x, the first difference series is

$$a(x-1),\quad ax(x-1),\quad ax^2(x-1), \ldots$$

which is a G.P. with the same common ratio.

Hence all the difference series are geometric progressions with common ratio x.

Also $\Delta^2 u_1 = a(x-1)^2, \quad \Delta^3 u_1 = a(x-1)^3, \ldots$

and (see p. 218), $\Delta^m u_1 = a(x-1)^m, \quad \Delta^m u_r = a(x-1)^m x^{r-1}$.

It is not however true that when the m^{th} difference series is a G.P., the original series must also be a G.P., and in fact it follows from p. 216 that if

$$u_r = ax^{r-1} + a_0 + a_1 r + a_2 r^2 + \ldots + a_k r^k,$$

then the $(k+1)^{\text{th}}$ difference series and all later difference series are G.P.'s. It therefore follows that if the $(k+1)^{\text{th}}$ difference series is

$$b, \quad bx, \quad bx^2, \quad bx^3, \ldots$$

a possible form of u_r is

$$bx^{r-1}/(x-1)^{k+1} + a_0 + a_1 r + a_2 r^2 + \ldots + a_k r^k,$$

and it is easy to prove that this is the most general form.

Example 16. Find by the method of differences the r^{th} term of a series whose first seven terms are 1, 2, 3, 6, 17, 54, 171, and find the sum to n terms.

The series and the successive difference series are

1	2	3	6	17	54	171 ...	
1	1	3	11	37	117		
0	2	8	26	80			
2	6	18	54				

Thus the determinate terms of the third difference series are terms of a G.P., and if we assume that $\Delta^3 u_r = 2.3^{r-1}$ for all values of r, it can be proved as stated above that

$$u_r = 2.3^{r-1}/2^3 + a_0 + a_1 r + a_2 r^2$$
$$= \tfrac{1}{4} 3^{r-1} + a + b(r-1) + c(r-1)(r-2) \quad \ldots\ldots\ldots\ldots\text{(i)}$$

By putting $r = 1, 2, 3$ in succession, we obtain

$$a = \tfrac{3}{4}, \quad b = \tfrac{1}{2}, \quad c = -\tfrac{1}{2},$$

hence $$u_r = \tfrac{1}{4}(3^{r-1} - 2r^2 + 8r - 3) \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

The sum to n terms can be found from (ii), but rather more easily from (i) which gives (see p. 40)

$$\tfrac{1}{4}(1+3+3^2+\ldots+3^{n-1})+an+\tfrac{1}{2}b(n-1)n+\tfrac{1}{3}c(n-2)(n-1)n,$$
$$=\tfrac{1}{8}(3^n-1)-\tfrac{1}{12}n(2n^2-9n-2).$$

Alternatively there exists as in Example 13, p. 215, a polynomial of degree six in r which takes the seven assigned values for $r=1, 2, 3, \ldots, 7$ and this can be obtained from Newton's formula by writing down three more of the difference series.

EXERCISE Xd

A

1. Calculate the first three difference series of the series

3, 13, 37, 81, 151, 253, 393.

2. Find by the method of differences the r^{th} term of the series 3, 7, 13, ... assuming it to be a quadratic in r.

3. Find by the method of differences the r^{th} term of the series 1, 7, 25, 61, ... assuming it to be a cubic in r.

In Nos. 4, 5, find the r^{th} term assuming it to be a polynomial in r of as low a degree as possible :

4. 4, 10, 18, ...

5. -4, 0, 6, ...

6. Evaluate $\sum_{1}^{n}(r^2+r+1)$ by the method of differences.

Find the r^{th} terms of the series in Nos. 7, 8, assuming them to be polynomials in r of as low a degree as possible. Find also the sums to n terms.

7. 1, 3, 7, 13, 21, ...

8. -1, 4, 21, 56, 115, ...

Find the r^{th} terms of the series in Nos. 9, 10, assuming that at the earliest possible stage the difference series are geometric progressions. Find also the sums to n terms.

9. 3, 4, 6, 10, 18, ...

10. 1, 11, 111, 1111, ...

B

11. Calculate the first four difference series of the series

$$7,\ 14,\ 31,\ 74,\ 191,\ 526,\ 1511.$$

12. Find the r^{th} term of the series 0, 5, 22, 57, ... assuming it to be a cubic in r.

13. Find the r^{th} term of the series 1, 2, 3, 4, 6, ... assuming it to be a polynomial of degree 4 in r.

In Nos. 14, 15, find the r^{th} terms assuming them to be polynomials in r of as low a degree as possible.

14. 3, 8, 17, ...

15. 0, 4, 18, 48, ...

16. Evaluate $\sum_{1}^{n}(2r^3 - 3r^2)$ by the method of differences.

Find the r^{th} terms of the series in Nos. 17, 18, assuming them to be polynomials in r of as low a degree as possible. Find also the sums to n terms.

17. 3, 15, 35, 63, ...

18. 1, 12, 45, 112, 225, 396, ...

Find the r^{th} terms of the series in Nos. 19, 20, assuming that at the earliest possible stage the difference series are geometric progressions. Find also the sums to n terms.

19. 1, 4, 10, ...

20. 4, 9, 17, 31, 57, ...

C

21. If $u_r = 1/r$, find $\Delta^n u_1$.

22. If $u_r = 1/\{r(r+1)\}$, find $\Delta^n u_1$.

In Nos. 23, 24, find the r^{th} terms assuming them to be polynomials in r of as low a degree as possible.

23. 0, 0, 12, 42, ...

24. 3, 4, 6, 10, ...

25. Evaluate $\sum_{1}^{n}(r-1)^2(r-2)^2$ by the method of differences.

Find the r^{th} terms of the series in Nos. 26, 27, assuming them to be polynomials in r of as low a degree as possible. Find also the sum to n terms.

26. 12, 50, 126, 252, 440, ...

27. 1, 2, 4, 8, 15, 26, 42, ...

Find the r^{th} terms of the series in Nos. 28, 29, assuming that at the earliest possible stage the difference series are geometric progressions. Find also the sums to n terms.

28. 3, 10, 27, 70, 187, ...

29. 7, 18, 46, 104, 212, 404, 742, ...

30. If $u_r = (-1)^{r-1}/r$, prove that $\Delta^n u_1 = (-1)^n(2^{n+1} - 1)/(n+1)$.

31. Prove that $\sum_1^n 1/r = \binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}$.

32. If $a_r = r/(r-x)$, prove that $x/(x-n)$ is the sum to $n+1$ terms of

$$1 - a_1\binom{n}{1} + a_1 a_2\binom{n}{2} - a_1 a_2 a_3\binom{n}{3} + \ldots$$

MISCELLANEOUS EXAMPLES

EXERCISE Xe

A

1. Find the coefficient of a^2b^2c in $(4a + 3b + 2c + d)^5$.

2. If n is a positive integer, prove that

$$(3n+1)\int_0^1 (1-x^3)^n dx = 3n\int_0^1 (1-x^3)^{n-1} dx,$$

and deduce that

$$1 - \frac{1}{4}\binom{n}{1} + \frac{1}{7}\binom{n}{2} - \ldots + (-1)^n \frac{1}{3n+1}\binom{n}{n} = \frac{3^n(n!)}{1.4.7\ldots(3n+1)}.$$

3. If $(\sqrt{2}+1)^{2m+1} = N + F$ where m, N are positive integers and $0 < F < 1$, prove that $N = 1/F - F$.

4. Prove that $\sum_1^n \frac{r+3}{r(r+1)(r+2)}\left(\frac{1}{3}\right)^r = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}\left(\frac{1}{3}\right)^n$.

5. Sum to n terms : $1.4.7 + 4.7.10 + 7.10.13 + \ldots$.

6. Sum to n terms : $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots$.

7. Sum to n terms the series $-1 + 6 + 37 + 140 + \ldots$ given that the second difference series is a G.P.

8. Prove that $\sum_0^n \binom{n}{r}^2$ is equal to the coefficient of $x^n y^n$ in the expansion of $\{(1+x)(1+y)(x+y)\}^n$.

9. Prove by evaluating $\frac{d^n}{dx^n}(x^{p+n}x^{-p-\frac{1}{2}})$ that

$$1 - \binom{n}{1}\frac{2p+1}{2p+2} + \binom{n}{2}\frac{(2p+1)(2p+3)}{(2p+2)(2p+4)} - \ldots \text{ to } n+1 \text{ terms}$$

equals $$\frac{1.3.5\ldots(2n-1)}{(2p+2)(2p+4)\ldots(2p+2n)}.$$

10. If $(1+x+x^2)^n \equiv \sum_0^{2n} a_r x^r$, prove that

$$a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2 = a_n.$$

B

11. Find the coefficient of x^7 in $(1-2x+x^3)^6$.

12. Find the sum to $n+1$ terms of

$$\frac{1}{2} - \frac{1}{3}\binom{n}{1} + \frac{1}{4}\binom{n}{2} - \frac{1}{5}\binom{n}{3} + \ldots$$

13. If $(1+px+x^2)^n \equiv \sum_0^{2n} a_r x^r$, prove that

(i) $1 + 3a_1 + 5a_2 + \ldots + (4n+1)a_{2n} = (2n+1)(2+p)^n$

(ii) $2 + 3a_1 + 5a_2 + \ldots + (2^{2n}+1)a_{2n} = (2+p)^n + (5+2p)^n$.

14. Find the sum to n terms and the sum to infinity of the series whose r^{th} term is $\dfrac{r+2}{r(r+1)}\left(\dfrac{1}{2}\right)^r$.

15. Prove that the sum to n terms of the series

$$\frac{1}{x+1} + \frac{1!}{(x+1)(x+2)} + \frac{2!}{(x+1)(x+2)(x+3)} + \ldots$$

is $\dfrac{1}{x} - \dfrac{n!}{x(x+1)\ldots(x+n)}$.

16. Sum to n terms : $1.5^2 + 5.9^2 + 9.13^2 + \ldots$

17. Sum to n terms : $\dfrac{1}{1.3.4} + \dfrac{1}{2.4.5} + \dfrac{1}{3.5.6} + \ldots$

18. Sum to n terms the series 8, 4, 2, 2, 4, ... , assuming that the r^{th} term is a polynomial in r of as low a degree as possible.

19. If $(5+2\sqrt{6})^m = N + F$, where m, N are positive integers and $0 < F < 1$, prove that $N = 1/(1-F) - F$.

20. The r^{th} polygonal number of order n is defined to be $r + \frac{1}{2}r(r-1)(n-2)$, $n > 2$. Prove that the sum of the first r of all sets of polygonal numbers of orders 2 to k inclusive is

$$\tfrac{1}{12}r(r+1)(k-1)(kr-2r-k+8).$$

C

21. Sum to n terms : $1-\frac{a}{1}+\frac{a(a-1)}{1.2}-\frac{a(a-1)(a-2)}{1.2.3}+\dots$

22. Sum to n terms : $\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+\dots$

23. If $u_r=\sin(\alpha+r\beta)$, prove that

$$\Delta^n u_r=(2\sin\tfrac{1}{2}\beta)^n\sin\{\alpha+\tfrac{1}{2}n(\pi+\beta)+r\beta\}.$$

24. If $n=mq+r$ where n, m, q, r are positive integers and $r<m$, prove that the greatest coefficient in the expansion of $(x_1+x_2+\dots+x_m)^n$ equals $n!/\{(q!)^m(q+1)^r\}$

25. If $(1+z)(1+xz)(1+x^2z)\dots(1+x^{n-1}z)\equiv 1+\sum_1^n a_r z^r$, prove that

(i) $a_r=a_{r-1}x^{r-1}(1-x^{n-r+1})/(1-x^r)$

(ii) $a_r=x^{\frac{1}{2}r(r-1)}\dfrac{(1-x^n)(1-x^{n-1})\dots(1-x^{n-r+1})}{(1-x)(1-x^2)\dots(1-x^r)}$.

26. Prove that the sum to r terms of the series

$$1-\frac{n}{m+1}+\frac{n(n-1)}{(m+1)(m+2)}-\frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}+\dots$$

equals $\{m/(m+n)\}\times$

$$\{1+(-1)^{r-1}[n(n-1)\dots(n-r+1)]/[m(m+1)\dots(m+r-1)]\}.$$

27. Prove that the sum of the reciprocals of the coefficients of x^r for $r=0$ to $2n$ in the expansion of $(1-x)^{2n}$ where n is a positive integer, is $(2n+1)/(n+1)$.

28. If $a\neq 1$, find the sum to n terms of

$$1+\frac{1+d}{a+2d}+\frac{(1+d)(1+2d)}{(a+2d)(a+3d)}+\frac{(1+d)(1+2d)(1+3d)}{(a+2d)(a+3d)(a+4d)}+\dots$$

29. If $s_r=1+2+3+\dots+r$, prove that

$$s_1s_n+s_2s_{n-1}+s_3s_{n-2}+\dots+s_ns_1=\tfrac{1}{120}n(n+1)(n+2)(n+3)(n+4).$$

30. Prove that the sum to $n+1$ terms of the series

$$1-\binom{n}{1}\frac{a}{b}+\binom{n}{2}\frac{a(a-1)}{b(b-1)}-\binom{n}{3}\frac{a(a-1)(a-2)}{b(b-1)(b-2)}+\dots$$

is $\{(b-a)(b-a-1)\dots(b-a-n+1)\}/\{b(b-1)\dots(b-n+1)\}$.

CHAPTER XI

DIFFERENCE EQUATIONS

Recurring Series. A method of summing the series

$$a+(a+d)x+(a+2d)x^2+\ldots+(a+rd)x^r+\ldots$$

is given on p. 44, and an alternative method of completing the work is indicated in Exercise IIId, No. 9, p. 46.

Consider for example the power series whose successive coefficients 2, 5, 8, ..., $3r-1$, ... are in A.P.

If $\quad S=2+5x+8x^2+\ldots+(3n-1)x^{n-1}$,

then $-2xS=-4x-10x^2-\ldots-2(3n-4)x^{n-1}-2(3n-1)x^n$,

and $\quad x^2S=2x^2+\ldots+(3n-7)x^{n-1}+(3n-4)x^n+(3n-1)x^{n+1}$.

But in the sum of these results the coefficient of x^{r-1} for $r=3, 4, \ldots, n$ is evidently $(3r-1)-2(3r-4)+(3r-7)$, that is 0,

∴ the sum (S) to n terms is given by

$$(1-2x+x^2)S=2+x-(3n+2)x^n+(3n-1)x^{n+1}.$$

This method owes its success to the fact that, if p_r denotes the coefficient of x^r,

$$p_r-2p_{r-1}+p_{r-2}=0, \quad (r=2, 3, \ldots, n-1)$$

and it can be used in all cases in which the coefficients are connected by such relations.

Example 1. Sum to n terms:

$$3+4x+6x^2+\ldots+(2+2^r)x^r+\ldots.$$

If $\quad S=3+4x+6x^2+10x^3+\ldots+(2+2^{n-1})x^{n-1}$,

then $axS=3ax+4ax^2+6ax^3+\ldots+(2+2^{n-2})ax^{n-1}+(2+2^{n-1})ax^n$,

and $bx^2S=3bx^2+4bx^3+\ldots+(2+2^{n-3})bx^{n-1}+(2+2^{n-2})bx^n+(2+2^{n-1})bx^{n+1}$.

Values can be chosen for a and b so that on addition the coefficients of x^2 and x^3 are zero. This requires that

$$6+4a+3b=0 \quad \text{and} \quad 10+6a+4b=0$$

and these equations give $a=-3$, $b=2$.

But for these values of a, b, the coefficient of x^r for

$$r=4, 5, \ldots, n-1 \text{ is } (2+2^r)-3(2+2^{r-1})+2(2+2^{r-2})$$

which equals $2^{r-1}(2-3+1)$, that is 0.

$\therefore$ the sum S to n terms is given by

$$S(1-3x+2x^2)=3-5x-(6+3.2^{n-1}-4-2^{n-1})x^n+(4+2^n)x^{n+1}.$$

$$\therefore \quad S=\{3-5x-(2+2^n)x^n+(4+2^n)x^{n+1}\}/(1-3x+2x^2).$$

In order that the method of Example 1 may be employed successfully for the general series

$$S \equiv p_0+p_1x+p_2x^2+\ldots+p_{n-1}x^{n-1},$$

it is necessary that after a and b have been chosen so that the coefficients of x^2 and x^3 in $S(1+ax+bx^2)$ are zero, the coefficients of x^r for $r=4, 5, \ldots, n-1$ should also be zero, that is

$$p_r+ap_{r-1}+bp_{r-2}=0, \quad (r=2, 3, 4, \ldots, n-1).$$

This equation in which a, b are constants is called a *linear difference equation* with *constant* coefficients, of *order* 2.

A similar method can be used if the coefficients are connected by the relation

$$p_r+a_1p_{r-1}+a_2p_{r-2}+a_3p_{r-3}=0$$

where a_1, a_2, a_3 are constants (see Exercise XIa, Nos. 31, 32), and this is called a linear difference equation with constant coefficients, of order 3.

The series

$$p_0, p_1, p_2, p_3, \ldots$$

whose terms satisfy the linear difference equation with constant coefficients, of order k,

$$p_r+a_1p_{r-1}+a_2p_{r-2}+\ldots+a_kp_{r-k}=0, \quad r>k,$$

is called a recurring series, and the difference equation is called the scale of relation of the series.

We shall return to the subject of recurring series after considering the solution of some difference equations.

The reader should now work Exercise XIa, Nos. 1-3.

Linear Difference Equations. We begin by considering equations of the second order with constant coefficients, i.e. equations of the form

$$u_r + au_{r-1} + bu_{r-2} = 0, \quad r \geqslant 3.$$

Since $\quad u_3 = -au_2 - bu_1 \quad$ and $\quad u_4 = -au_3 - bu_2,$

$$u_4 = a(au_2 + bu_1) - bu_2 = abu_1 + (a^2 - b)u_2.$$

Therefore u_3, u_4 can each be expressed in the form $\lambda u_1 + \mu u_2$ where λ, μ are independent of u_1 and u_2. It follows by induction that u_r can also be expressed in this form, so that

$$u_r = \lambda_r u_1 + \mu_r u_2$$

where λ_r, μ_r are functions of r but are independent of u_1, u_2.

Arbitrary values can be assigned to u_1, u_2, and therefore the general solution of the equation

$$u_r + au_{r-1} + bu_{r-2} = 0$$

contains two arbitrary constants which occur in a linear form.

It is characteristic of all linear equations and is evident by direct substitution that if $u_r = f(r)$ and $u_r = g(r)$ are two solutions, and if A, B are any two numbers independent of r, then $u_r = Af(r) + Bg(r)$ is also a solution; this contains two arbitrary constants if $f(r)$ and $g(r)$ are not proportional, i.e. if their ratio is not independent of r, and is the general solution. If however $f(r)$ and $g(r)$ are proportional, $Af(r) + Bg(r)$ can be expressed in the form $(AA_1 + BB_1)h(r)$, $\equiv C\,h(r)$, where A_1, B_1, C are independent of r, and therefore involves only one arbitrary constant.

It can be shown similarly that a third order difference equation with constant coefficients

$$u_r + au_{r-1} + bu_{r-2} + cu_{r-3} = 0, \quad r \geqslant 4,$$

has a general solution of the form

$$u_r = Af(r) + Bg(r) + Ch(r)$$

where A, B, C are arbitrary constants.

Also the general k^{th} order difference equation with constant coefficients has a general solution

$$u_r = C_1\alpha_1 + C_2\alpha_2 + \dots + C_k\alpha_k$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are independent particular solutions and $C_1, C_2, \dots, C_k$ are arbitrary constants.

Note. $\alpha_1, \alpha_2, \dots, \alpha_k$ are called *independent* if it is impossible to find constants $A_1, A_2, \dots, A_k$, not all zero, such that

$$A_1\alpha_1 + A_2\alpha_2 + \dots + A_k\alpha_k = 0.$$

We now proceed to indicate by examples the methods of finding particular independent solutions and hence the general solution. The methods apply to equations of any order.

The reader who is familiar with the solution of linear differential equations of the second order with constant coefficients will recognise the analogies between difference equations and differential equations (see *Durell and Robson :* Elem. Calculus, Vol. II, pp. 410-413).

Example 2. Solve the difference equation

$$u_r - 5u_{r-1} + 6u_{r-2} = 0, \quad r \geqslant 3.$$

If $u_r = x^r$, then $u_{r-1} = x^{r-1}$ and $u_{r-2} = x^{r-2}$. Hence the equation is satisfied if

$$x^r - 5x^{r-1} + 6x^{r-2} = 0,$$

that is if $x^2 - 5x + 6 = 0$, which gives $x = 2$ or 3.

Thus $u_r = 2^r$, $u_r = 3^r$, are two particular solutions. As they are independent, the general solution is

$$u_r = A\,2^r + B\,3^r$$

where A, B are arbitrary constants.

Note. The solution of Example 2 might equally well be taken in the form $u_r = C\,2^{r-1} + D\,3^{r-1}$ where C, D are arbitrary constants. Sometimes the use of this form effects a considerable saving in numerical work.

Example 3. Find u_r if $u_1 = 8$, $u_2 = 200$, and

$$u_r = 63u_{r-2} - 2u_{r-1}, \quad r \geqslant 3.$$

As in Example 2 the equation is satisfied by

$$u_r = x^r \text{ if } x^2 + 2x - 63 = 0, \text{ i.e. if } x = 7 \text{ or } -9.$$

Hence $\qquad u_r = C\,7^{r-1} + D\,(-9)^{r-1}.$

Put $r = 1$; then $\qquad C + D = u_1 = 8.$

Put $r = 2$; then $\qquad 7C - 9D = u_2 = 200.$

Thus $\qquad C = 17, \quad D = -9, \quad u_r = 17.7^{r-1} + (-9)^r.$

(If $u_r = A7^r + B(-9)^r$ had been used, the equations for A, B would have been $7A - 9B = 8$ and $49A + 81B = 200$.)

Example 4. Solve the difference equation

$$u_r - 8u_{r-1} + 16u_{r-2} = 0, \quad r \geqslant 3.$$

If $u_r = x^r$, we have as in Examples 2, 3,

$$x^2 - 8x + 16 = 0, \quad \therefore (x-4)^2 = 0, \quad \therefore x = 4.$$

Therefore $u_r = A4^r$ is a solution, but as it contains only one arbitrary constant it is not the general solution.

Put $u_r = 4^r v_r$; then $4^r v_r - 8.4^{r-1} v_{r-1} + 16.4^{r-2} v_{r-2} = 0.$

$$\therefore v_r - 2v_{r-1} + v_{r-2} = 0$$

$\therefore$ by p. 218, $\Delta^2 v_{r-2} = 0$ and so $\Delta^2 v_r = 0$.

$\therefore$ by p. 216, $v_r = B + Cr$ where B, C are arbitrary constants.

Hence $\qquad u_r = (B + Cr)4^r$

and this is the general solution because it contains two arbitrary constants.

The reader should now work Exercise XIa, Nos. 4-10.

Example 5. Solve the difference equation

$$u_{r+3} - 4u_{r+2} + u_{r+1} + 6u_r = 0, \quad r \geqslant 1.$$

Put $u_r = x^r$, then the equation is satisfied if $x^3 - 4x^2 + x + 6 = 0$, that is if $(x+1)(x-2)(x-3) = 0$.

Hence the general solution containing three arbitrary constants is

$$u_r = (-1)^r A + 2^r B + 3^r C.$$

Example 6. Solve the difference equation

$$u_{r+3} - 6u_{r+2} + 12u_{r+1} - 8u_r = 0, \quad r > 1.$$

Put $u_r = x^r$, then the equation is satisfied if $x^3 - 6x^2 + 12x - 8 = 0$, that is if $(x-2)^3 = 0$, $\therefore$ $x = 2$.

Hence $u_r = 2^r$ is a solution. To obtain the general solution put $u_r = 2^r v_r$, then $8v_{r+3} - 24v_{r+2} + 24v_{r+1} - 8v_r = 0$,

$$\therefore \; v_{r+3} - 3v_{r+2} + 3v_{r+1} - v_r = 0.$$

$\therefore$ by p. 218, $\qquad \Delta^3 v_r = 0.$

$\therefore$ by p. 216, $v_r = A + Br + Cr^2$ where A, B, C are arbitrary constants.

Hence the general solution is $\; u_r = 2^r(A + Br + Cr^2)$.

Example 7. Solve the difference equation

$$2u_{r+4} - 15u_{r+3} + 42u_{r+2} - 52u_{r+1} + 24u_r = 0, \quad r > 1.$$

Put $u_r = x^r$. Then the equation is satisfied if

$$2x^4 - 15x^3 + 42x^2 - 52x + 24 = 0,$$

that is if

$$(x-2)^3(2x-3) = 0.$$

Hence $u_r = 2^r$ and $u_r = (\frac{3}{2})^r$ are solutions. The work of Example 6 suggests that $u_r = 2^r(A + Br + Cr^2)$ is also a solution. This can be proved as follows.

The factorisation $(2x-3)(x^3 - 6x^2 + 12x - 8)$ suggests writing the difference equation in the form

$$2(u_{r+4} - 6u_{r+3} + 12u_{r+2} - 8u_{r+1}) - 3(u_{r+3} - 6u_{r+2} + 12u_{r+1} - 8u_r) = 0,$$

which shows that any value of u_r which satisfies

$$u_{r+3} - 6u_{r+2} + 12u_{r+1} - 8u_r = 0$$

and which therefore satisfies

$$u_{r+4} - 6u_{r+3} + 12u_{r+2} - 8u_{r+1} = 0$$

also satisfies the given difference equation.

Hence from Example 6, $u_r = 2^r(A + Br + Cr^2)$ is a solution.

Thus the general solution, containing four arbitrary constants, is

$$u_r = 2^r(A + Br + Cr^2) + (\tfrac{3}{2})^r D.$$

The case in which the corresponding equation for x has complex roots is illustrated in Example 8.

Example 8. Solve the difference equation

$$u_r - 8u_{r-1} + 25u_{r-2} = 0, \quad r \geqslant 2.$$

Put $u_r = x^r$. Then the equation is satisfied if $x^2 - 8x + 25 = 0$, which gives $x = 4 + 3i$ or $4 - 3i$.

Hence $$u_r = A(4+3i)^r + B(4-3i)^r.$$

Choose ρ, α so that $\rho \cos \alpha = 4$, $\rho \sin \alpha = 3$, $\rho > 0$; then $\rho = 5$, and α is given by $\cos \alpha : \sin \alpha : 1 = 4 : 3 : 5$, and

$$u_r = A\rho^r \operatorname{cis} r\alpha + B\rho^r \operatorname{cis}(-r\alpha)$$

where $\operatorname{cis} \theta$ denotes $\cos \theta + i \sin \theta$.

Hence $$u_r = 5^r\{(A+B)\cos r\alpha + i(A-B)\sin r\alpha\}$$

which may be expressed in the form

$$u_r = 5^r(C \cos r\alpha + D \sin r\alpha)$$

where C, D are arbitrary constants.

The reader should now work Exercise XIa, Nos. 11-14.

We shall not discuss the solution of linear difference equations with variable coefficients, but Example 9 illustrates a method which can sometimes be used successfully. See Exercise XIa, Nos. 15, 16, 38-40.

Example 9. Find u_n if $u_1 = 0$, $u_2 = 1$ and

$$u_{r+1} = r(u_r + u_{r-1}), \quad r \geqslant 2.$$

The equation may be written

$$u_{r+1} - (r+1)u_r = -\{u_r - ru_{r-1}\}.$$

Similarly $$u_r - ru_{r-1} = -\{u_{r-1} - (r-1)u_{r-2}\},$$

and so on.

Hence $$u_{r+1} - (r+1)u_r = (-1)^2\{u_{r-1} - (r-1)u_{r-2}\} = \dots$$
$$= (-1)^{r-1}(u_2 - 2u_1) = (-1)^{r-1}.$$

Dividing by $(r+1)!$, we have

$$\frac{u_{r+1}}{(r+1)!}-\frac{u_r}{r!}=\frac{(-1)^{r+1}}{(r+1)!}$$

and putting 1, 2, 3, ..., $n-1$ in succession for r and adding, we obtain

$$\frac{u_n}{n!}-\frac{u_1}{1!}=\sum_2^n\frac{(-1)^r}{r!}$$

$$\therefore\ u_n=(n!)\left\{\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right\}.$$

The series within the bracket cannot be summed, but for large values of r its value is approximately $1/e$.

EXERCISE XIa

A

1. Find the numerical values of a, b such that the expansion of

$$(1+ax+bx^2)\{3+5x+7x^2+\ldots+(2n+1)x^{n-1}\}$$

involves no terms in $x^2, x^3, \ldots, x^{n-1}$. Hence find the sum of the series in the second bracket.

Use the method of No. 1 to find the sum to n terms of the series in Nos. 2, 3.

2. $1-4x+7x^2-10x^3+\ldots+(-1)^r(3r+1)x^r+\ldots$

3. $1-3x+7x^2-15x^3+\ldots+(-1)^r(2^{r+1}-1)x^r+\ldots$

Solve the difference equations in Nos. 4, 5.

4. $u_{r+2}+u_{r+1}-12u_r=0,\quad r>1.$

5. $u_r-10u_{r-1}+25u_{r-2}=0,\quad r>3.$

Find difference equations with numerical coefficients, whose general solutions are as follows:

6. $u_r=A+Br+Cr^2.$ 7. $u_r=3^rA+7^rB.$

Find u_r in Nos. 8-10.

8. $u_r+3u_{r-1}-4u_{r-2}=0,\ r>3;\ u_1=21,\ u_2=1.$

9. $u_r-6u_{r-1}+9u_{r-2}=0,\ r>2;\ u_0=5,\ u_1=9.$

10. $u_r-u_{r-1}=a,\ r>1;\ u_0=2a.$

11. Solve the difference equation $u_r - 2u_{r-1} - u_{r-2} + 2u_{r-3} = 0$, $r > 4$.

Find u_r in Nos. 12-14.

12. $u_{r+3} - 6u_{r+2} + 11u_{r+1} - 6u_r = 0$, $r > 1$; $u_1 = 2$, $u_2 = 6$, $u_3 = 20$.

13. $u_r - 12u_{r-2} + 16u_{r-3} = 0$, $r > 3$; $u_0 = 4$, $u_1 = -8$, $u_2 = -12$.

14. $u_r - 6u_{r-1} + 25u_{r-2} = 0$, $r > 3$; $u_1 = 1$, $u_2 = 0$

Find expressions for u_r in Nos. 15, 16.

15. $u_r - ru_{r-1} = r - 1$, $r > 2$; $u_1 = 0$

16. $u_r = (r+1)u_{r-1} - ru_{r-2}$, $r > 3$; $u_1 = 1$, $u_2 = 3$.

B

17. Find the numerical values of a, b such that the coefficients of x^2 and x^3 in the expansion of $(1 + ax + bx^2)(1 + 4x + 7x^2 + 10x^3)$ are zero. Hence find the 5th term of the recurring series of order 2 whose first four terms are 1, 4, 7, 10.

Sum to n terms the series in Nos. 18, 19.

18. $1 + 2x + 5x^2 + 14x^3 + \dots + \frac{1}{2}(3^r + 1)x^r + \dots$

19. $1 + 4x + 14x^2 + 46x^3 + \dots + (2.3^r - 2^r)x^r + \dots$

Solve the difference equations in Nos. 20, 21.

20. $u_r - 6u_{r-1} + 8u_{r-2} = 0$, $r > 3$

21. $u_r + 3u_{r-1} + 2u_{r-2} = 0$, $r > 3$

Find difference equations with numerical coefficients, whose general solutions are as follows:

22. $u_r = A + Br$. 23. $u_r = A + 4^r B$. 24. $u_r = A(-1)^r + B(\frac{1}{2})^r$.

Find u_r in Nos. 25, 26.

25. $u_r + 2u_{r-1} + u_{r-2} = 0$, $r > 3$; $u_1 = 1$, $u_2 = 0$.

26. $2u_{r+2} + 2u_r = 5u_{r+1}$, $r > 1$; $u_1 = 4$, $u_2 = 5$.

27. Solve the difference equation $u_r - 7u_{r-2} + 6u_{r-3} = 0$, $r > 4$.

Find u_r in Nos. 28, 29.

28. $u_r - 13u_{r-2} + 12u_{r-3} = 0$, $r > 4$; $u_1 = -7$, $u_2 = -8$, $u_3 = 24$.

29. $u_r - 2u_{r-1} + 2u_{r-2} = 0$, $r > 2$; $u_0 = 0$, $u_1 = 1$.

30. Find an expression for u_r if $ru_r - u_{r-1} = r$, $r > 2$; $u_1 = 1$.

C

31. Find the numerical values of a, b, c such that the coefficients of x^3, x^4, x^5 in the expansion of

$$(1 + ax + bx^2 + cx^3)(1 + x + 3x^2 + 7x^3 + 13x^4 + 21x^5)$$

are zero. Hence find the 7th term of the recurring series of order 3 whose first six terms are 1, 1, 3, 7, 13, 21.

32. Use the method of No. 31 to sum to n terms the series

$$1 + 3x + 7x^2 + \ldots + (r^2 + r + 1)x^r + \ldots$$

33. Find the difference equation with numerical coefficients, whose general solution is $u_r = (A + Br)6^r$.

34. Find u_r if $u_r = (\alpha + \beta)u_{r-1} - \alpha\beta u_{r-2}$, $r > 3$; $u_1 = 1/\alpha$, $u_2 = \beta$.

Solve the difference equations in Nos. 35, 36.

35. $u_r - 9u_{r-1} + 27u_{r-2} - 27u_{r-3} = 0$, $r > 4$.

36. $u_{r+3} - 5u_{r+2} + 7u_{r+1} - 3u_r = 0$, $r > 1$.

37. Find u_r if $u_r - 4u_{r-1} + 6u_{r-2} - 4u_{r-3} + u_{r-4} = 0$, $r > 4$, given that $u_1 = u_2 = -u_3 = u_4 = 1$.

Find expressions for u_r in Nos. 38-40.

38. $u_r = (r + 1)u_{r-1} - (r - 1)u_{r-2}$, $r > 3$; $u_1 = u_2 = 1$.

39. $u_r = ru_{r-1} + (3r + 6)u_{r-2}$, $r > 3$; $u_1 = 1$, $u_2 = 5$.

40. $u_r = (r + 1)(u_{r-1} - u_{r-2})$, $r > 3$, if

(i) $u_1 = 2$, $u_2 = 3$, (ii) $u_1 = 0$, $u_2 = 3$.

Recurring Series continued. The methods for solving linear difference equations can be applied to the problem of determining the general term of a recurring series whose scale of relation (see p. 227) is given.

Example 10. If the scale of relation of the recurring series

$$1,\ 2,\ 1,\ -22,\ \ldots$$

is $u_r + au_{r-1} + bu_{r-2} = 0$, find the values of a, b, and u_r.

$$u_3 + au_2 + bu_1 = 0, \qquad \therefore\ 1 + 2a + b = 0\ ;$$

$$u_4 + au_3 + bu_2 = 0, \qquad \therefore\ -22 + a + 2b = 0\ ;$$

$$\therefore\ a = -8,\ b = 15\ ; \qquad \therefore\ u_r - 8u_{r-1} + 15u_{r-2} = 0.$$

Hence as in Example 2, p. 229, $u_r = x^r$ is a solution if

$$x^2 - 8x + 15 = 0,$$

which gives $x = 3$ or 5.

$$\therefore\ u_r = 3^{r-1}A + 5^{r-1}B$$

where $\qquad A + B = u_1 = 1$ and $3A + 5B = u_2 = 2,$

$$\therefore\ A = \tfrac{3}{2},\ B = -\tfrac{1}{2} \qquad \therefore\ u_r = \tfrac{1}{2}(3^r - 5^{r-1}).$$

There is an unlimited number of recurring series having the same first four terms as the series in Example 10, but their scales of relation are of higher orders. Thus the reader may verify that the series given by

$$70u_r = 7(-1)^r + 40.2^r - 3^{2r-1}$$

whose scale of relation (order 3) is $u_r - 10u_{r-1} + 7u_{r-2} + 18u_{r-3} = 0$, also has 1, 2, 1, -22 for its first four terms.

It may also be verified that the method of Example 12, p. 214, gives $u_r = -\frac{1}{3}(10r^3 - 57r^2 + 98r - 54)$ which is the general term of a recurring series of order 4, and that the second process of Example 13, p. 215, gives $u_r = -\frac{1}{50}(11^{r-1} - 60r + 9)$ which is also the general term of a recurring series of order 4.

Power Series. The series

$$u_0 + u_1x + u_2x^2 + \ldots + u_rx^r + \ldots$$

whose coefficients form a recurring series is called a **recurring power series**, and the scale of relation of the *coefficients* is called the *scale of relation of the power series.*

The method on p. 226 for finding the sum of n terms of a recurring series can be used to show that a recurring power series in x is, for all values of x for which it is convergent, the expansion of a rational function of x in which the degree of the numerator is less than that of the denominator. For simplicity we consider a series of order 2, but the method is general.

Let the scale of relation of the power series

$$u_0 + u_1 x + u_2 x^2 + \dots + u_r x^r + \dots$$

be

$$u_r + a_1 u_{r-1} + a_2 u_{r-2} = 0, \quad r \geqslant 2,$$

and denote the sum to n terms by S_n.

$$S_n = u_0 + u_1 x + u_2 x^2 + \dots + u_{n-1} x^{n-1}$$

$$\therefore\ a_1 x S_n = a_1 u_0 x + a_1 u_1 x^2 + \dots + a_1 u_{n-2} x^{n-1} + a_1 u_{n-1} x^n$$

and

$$a_2 x^2 S_n = \qquad a_2 u_0 x^2 + \dots + a_2 u_{n-3} x^{n-1} + a_2 u_{n-2} x^n + a_2 u_{n-1} x^{n+1}$$

If these results are added, the coefficient of x^r for $r = 2, 3, \dots, n-1$, is $u_r + a_1 u_{r-1} + a_2 u_{r-2}$, i.e. zero.

$$\therefore\ (1 + a_1 x + a_2 x^2) S_n = u_0 + (u_1 + a_1 u_0) x + (a_1 u_{n-1} + a_2 u_{n-2}) x^n + a_2 u_{n-1} x^{n+1}.$$

Now for any value of x for which the given power series is convergent,

$$\lim_{n\to\infty} u_{n-1} x^{n-1} = 0 \quad \text{(see p. 64)}$$

hence also $\lim_{n\to\infty} \{(a_1 u_{n-1} + a_2 u_{n-2}) x^n + a_2 u_{n-1} x^{n+1}\} = 0,$

$$\therefore\ \lim_{n\to\infty} S_n = \frac{u_0 + (u_1 + a_1 u_0) x}{1 + a_1 x + a_2 x^2}$$

for all values of x for which the given power series is convergent.

Thus, for this range of values, the power series is the expansion of the rational function $\dfrac{u_0 + (u_1 + a_1 u_0) x}{1 + a_1 x + a_2 x^2}$ which is called the **generating function** of the recurring power series.

The expression of the generating function in partial fractions provides a method of expansion in powers of x, from which the general term of the recurring power series can be found. This is therefore an alternative to the method used in Example 10, p. 236.

Example 11. Find the generating function and the r^{th} term of the recurring power series of order 2

$$3 + 4x + 6x^2 + 10x^3 + \dots$$

By hypothesis the scale of relation is of the form

$$u_r + au_{r-1} + bu_{r-2} = 0, \quad r > 2.$$

Here $\qquad 6 + 4a + 3b = 0$ and $10 + 6a + 4b = 0$,

$\therefore$ $a = -3$, $b = 2$, and the scale of relation is $u_r - 3u_{r-1} + 2u_{r-2} = 0$.

Also if $\qquad S_n = 3 + 4x + 6x^2 + \dots + u_{n-2}x^{n-2} + u_{n-1}x^{n-1}$,

$$(1 - 3x + 2x^2)S_n = 3 - 5x + x^n(-3u_{n-1} + 2u_{n-2}) + 2u_{n-1}x^{n+1};$$

$$\therefore \text{ the generating function} = \frac{3 - 5x}{1 - 3x + 2x^2} = \frac{2}{1-x} + \frac{1}{1-2x}$$

$\therefore$ if $|x| < \frac{1}{2}$, the generating function is the sum to infinity of

$$2(1 + x + x^2 + \dots + x^{r-1} + \dots) + (1 + 2x + 2^2x^2 + \dots + 2^{r-1}x^{r-1} + \dots),$$

$\therefore$ the r^{th} term of the recurring power series is $(2 + 2^{r-1})x^{r-1}$.

It is suggested that the reader should obtain this result by using the method of Example 10, p. 236. It will be found that there is little to choose between the two methods on the score of length.

Examples 10, 11 illustrate the fact that if only the first 4 terms of a recurring series are given, a scale of relation of order 2 can be found. If the first $2k - 1$ terms or the first $2k$ terms are given, it is not in general possible to determine a scale of relation of lower order than k, and if $2k$ terms are given a scale of relation of order k can in general be determined uniquely.

If the scale of relation is of order k, the denominator of the generating function is of the k^{th} degree. In particular if u_r is a polynomial of degree $k - 1$ in r, it follows from p. 213 that $\Delta^k u_r = 0$ and therefore from p. 218 that the scale of relation is

$$u_{r+k} - \binom{k}{1} u_{r+k-1} + \binom{k}{2} u_{r+k-2} - \dots + (-1)^k u_r = 0,$$

and hence the denominator of the generating function is $(1 - x)^k$.

Example 12. Find a scale of relation, the corresponding r^{th} term, and the sum to n terms of the recurring power series

$$2 + 5x + 10x^2 + 19x^3 + 36x^4 + \dots$$

Since 5 terms are given, it is in general impossible to find a scale of relation of lower order than 3. Let the relation be

$$u_r + au_{r-1} + bu_{r-2} + cu_{r-3} = 0$$

Then $\quad 19 + 10a + 5b + 2c = 0$ and $36 + 19a + 10b + 5c = 0,$

and these equations are insufficient to determine a, b, c.

If we choose arbitrarily $c = -2$, we get $a = -4$, $b = 5$, and the corresponding scale of relation is

$$u_r - 4u_{r-1} + 5u_{r-2} - 2u_{r-3} = 0.$$

We then find

$$\begin{aligned}(1 - 4x + 5x^2 - 2x^3)S_n &= (1 - 4x + 5x^2 - 2x^3)(2 + 5x + 10x^2 + 19x^3 + \dots + u_{n-1}x^{n-1})\\ &= 2 - 3x + x^n(-4u_{n-1} + 5u_{n-2} - 2u_{n-3})\\ &\qquad + x^{n+1}(5u_{n-1} - 2u_{n-2}) - 2u_{n-1}x^{n+2}\end{aligned}$$

$$\therefore \text{ the generating function} = \frac{2 - 3x}{1 - 4x + 5x^2 - 2x^3} = \frac{2 - 3x}{(1 - x)^2(1 - 2x)}$$

$$= \frac{1}{(1 - x)^2} - \frac{1}{1 - x} + \frac{2}{1 - 2x}.$$

Hence if $|x| < \frac{1}{2}$, the generating function is the sum to infinity of

$$(1 + 2x + \dots + rx^{r-1} + \dots) - (1 + x + \dots + x^{r-1} + \dots)$$
$$+ 2(1 + 2x + \dots + 2^{r-1}x^{r-1} + \dots)$$

and so the r^{th} term is $(r - 1 + 2^r)x^{r-1}$.

In the expression for $(1 - 4x + 5x^2 - 2x^3)S_n$, the coefficient of x^n

$$= -4u_{n-1} + 5u_{n-2} - 2u_{n-3} = -u_n = -(n + 2^{n+1}),$$

and the coefficient of x^{n+1}

$$\begin{aligned}&= 5u_{n-1} - 2u_{n-2}\\ &= 5(n - 1 + 2^n) - 2(n - 2 + 2^{n-1}) = 3n - 1 + 2^{n+2},\end{aligned}$$

$$\therefore \; S_n = (2 - 3x + R_n)/(1 - 4x + 5x^2 - 2x^3)$$

where $\; R_n = -(n + 2^{n+1})x^n + (3n - 1 + 2^{n+2})x^{n+1} - 2(n - 1 + 2^n)x^{n+2}$

Note. We can choose for c any other value *except* zero, and then obtain a different scale of relation corresponding to another power series starting with the 5 given terms. But we cannot take $c=0$, because the scale of relation would then become

$$u_r + au_{r-1} + bu_{r-2} = 0,$$

and unless we disregard the first term, this involves the incompatible equations

$$10 + 5a + 2b = 0, \quad 19 + 10a + 5b = 0, \quad 36 + 19a + 10b = 0.$$

If however a sixth term of the series in Example 12 had been given, the scale of relation of order 3 would have been determined uniquely.

Use of Complex Numbers. Many series are best summed by introducing complex numbers. The method is explained in the authors' *Advanced Trigonometry* to which the reader is referred. Thus although the sum to n terms of the series

$$1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^r\cos r\theta + \ldots$$

may be found by showing that it is a recurring power series whose scale of relation is $u_r - 2\cos\theta\, u_{r-1} + u_{r-2} = 0$, it is simpler to use De Moivre's theorem (*Adv. Trig.*, p. 174).

EXERCISE XIb

A

The series in Nos. 1-3 are recurring series of order 2. Find the r^{th} term and the sum to n terms.

1. $-1, 0, 12, 84, \ldots$ 2. $1, 3, 11, 43, \ldots$ 3. $\frac{1}{2}, 2, 5, 11, \ldots$

The series in Nos. 4-6 are recurring power series of order 2. Find the generating function and the coefficient of x^r.

4. $1 + 2x + 5x^2 + 14x^3 + \ldots$ 5. $2 + 5x + 8x^2 + 11x^3 + \ldots$

6. $1 - 8x + 28x^2 - 80x^3 + \ldots$

7. Find the generating function of the recurring power series $1 + 3x + 2x^2 - x^3 - 3x^4 + \ldots$, choosing a scale of relation of as low an order as possible.

8. Sum to n terms the recurring power series

$$2 + 8x + 18x^2 + 37x^3 + \ldots \text{ of order 2.}$$

Find the generating functions of the recurring power series of order 2 in Nos. 9, 10, and find the coefficient of x^r by using De Moivre's theorem.

9. $1 + 2x\sqrt{2} + 3x^2 + x^3\sqrt{2} + \ldots$

10. $\sin\theta + x\sin 2\theta + x^2\sin 3\theta + x^3\sin 4\theta + \ldots$

B

The series in Nos. 11-13 are recurring series of order 2. Find the r^{th} term and the sum to n terms.

11. 5, 12, 30, 78, ... 12. 4, 1, 7, −5, ... 13. 1, 6, 40, 288, ...

The series in Nos. 14-16 are recurring power series of order 2. Find the generating function and the coefficient of x^r.

14. $2 + 6x + 20x^2 + 72x^3 + \ldots$ 15. $5 + 19x + 83x^2 + 391x^3 + \ldots$

16. $5 - 2x + 8x^2 + 4x^3 + \ldots$

17. Find the generating function of the recurring power series $1 + 3x + 8x^2 + 20x^3 + 49x^4 + 119x^5 + \ldots$, choosing a scale of relation of as low an order as possible.

18. Find the sum to n terms of the recurring power series $2 + 8x + 34x^2 + 152x^3 + \ldots$ of order 2.

19. Find the generating function of the recurring power series $1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \ldots$ of order 2 and find the coefficient of x^r by using De Moivre's theorem.

C

20. Find the generating function of the recurring power series $1 + 2x + 3x^2 + 8x^3 + 9x^4 + \ldots$, choosing a scale of relation of as low an order as possible.

Find the generating functions of the recurring power series in Nos. 21, 22.

21. $4 + 4x + 2x^2 - 8x^3 - 46x^4 - 176x^5 + \ldots$, order 3.

22. $4 - 9x + 5x^2 - 19x^3 - 3x^4 - 47x^5 + \ldots$, order 3.

Find the generating functions of the recurring power series of order 2 whose first four terms are given in Nos. 23, 24, and use De Moivre's theorem to find the coefficient of x^r.

23. $8 - 12x - 8x^2 - x^3 + \ldots$

24. $\cos\theta + x\cos 3\theta + x^2\cos 5\theta + x^3\cos 7\theta + \ldots$

25. Obtain the expansion of $1/(1 + x + x^2)$ in powers of x by solving the difference equation $p_r + p_{r-1} + p_{r-2} = 0$, $r > 2$, $p_0 = 1$, $p_1 = -1$. Verify the result by another method.

Continued Fractions. Sometimes it is desirable to evaluate a fraction such as

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \dots + \cfrac{b_n}{a_n}}}}$$

from the top instead of by the usual arithmetical process from the bottom. For convenience the fraction is written

$$a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \frac{b_4}{a_4 +} \dots \frac{b_n}{+ a_n}.$$

a_1, $a_1 + \frac{b_2}{a_2}$, $a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3}$, ... may be regarded as approximations to the value of the continued fraction. They are called the 1st, 2nd, 3rd, ... *convergents.*

It will be shown in Example 16 that convergents can sometimes be calculated by solving a linear difference equation, but before explaining this method we shall show by an example how a rational number may be expressed in the form

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \dots \frac{1}{+ a_n}$$

where a_r is a positive integer except that a_1 may be zero.

A fraction of this form is called a *unit continued fraction.*

Example 13. Express $\frac{13}{31}$ as a unit continued fraction and calculate the convergents.

$$\frac{31}{13} = 2 + \frac{5}{13} \quad \frac{13}{5} = 2 + \frac{3}{5} \quad \frac{5}{3} = 1 + \frac{2}{3} \quad \frac{3}{2} = 1 + \frac{1}{2}$$

hence $$\frac{13}{31} = \frac{1}{2 +} \frac{1}{2 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{2}.$$

The convergents are $\frac{1}{2}$, $\frac{1}{2\frac{1}{2}}=\frac{2}{5}$, $\frac{1}{2+}\frac{1}{2+}\frac{1}{1}=\frac{1}{2+}\frac{1}{3}=\frac{3}{7}$

$$\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1}=\frac{1}{2+}\frac{1}{2+}\frac{1}{2}=\frac{1}{2+}\frac{2}{5}=\frac{5}{12}.$$

In this example the approximations to $\frac{13}{31}$ are $\frac{1}{2}, \frac{2}{5}, \frac{3}{7}, \frac{5}{12}$ and these are alternately too great and too small, because the denominators of the curtailed expressions $\frac{1}{2}$, $\frac{1}{2+}\frac{1}{2}$, ... are alternately less and greater than the denominator of $\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$.

This method is applicable to any rational number and the expression of such a number in the form of a unit continued fraction is unique.

Example 14. Express $\sqrt{6}$ in the form $a+\frac{1}{b+}\frac{1}{c+}\frac{1}{x}$ where a, b, c are positive integers.

$$\sqrt{6}=2+(\sqrt{6}-2)=2+1/p, \text{ where}$$
$$p=1/(\sqrt{6}-2)=(\sqrt{6}+2)/2=2+\tfrac{1}{2}(\sqrt{6}-2)=2+1/(2p),$$

hence
$$\sqrt{6}=2+\frac{1}{2+}\frac{1}{2p}$$
$$=2+\frac{1}{2+}\frac{1}{4+}\frac{1}{p}, \text{ where } p=\tfrac{1}{2}(\sqrt{6}+2).$$

By substituting the value $2+1/(2p)$ for p in the result of this example, it is found that

$$\sqrt{6}=2+\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{2p}$$

and substituting again

$$\sqrt{6}=2+\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\frac{1}{p};$$

and so on. We shall show on p. 361 that $\sqrt{6}$ may be regarded as the value of the infinite continued fraction

$$2+\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\cdots.$$

It is evident that $\sqrt{6}$ cannot be expressed in the form of a terminated unit continued fraction since elementary arithmetic shows that such a fraction is rational.

Calculation of Successive Convergents.

Let $\quad F_1=a_1,\ F_2=a_1+\dfrac{1}{a_2},\ F_3=a_1+\dfrac{1}{a_2+}\dfrac{1}{a_3}$, etc.

Then $\quad F_1=\dfrac{p_1}{q_1}$ if $p_1=a_1,\ q_1=1$;

$$F_2=\frac{a_1a_2+1}{a_2}=\frac{p_2}{q_2} \text{ if } p_2=a_1a_2+1,\ q_2=a_2;$$

F_3 may be found from F_2 by replacing a_2 by $a_2+\dfrac{1}{a_3}$,

thus $$F_3=\frac{a_1\left(a_2+\dfrac{1}{a_3}\right)+1}{a_2+\dfrac{1}{a_3}}=\frac{a_3(a_1a_2+1)+a_1}{a_3a_2+1}=\frac{p_3}{q_3}$$

if $p_3=a_3(a_1a_2+1)+a_1,\ q_3=a_3a_2+1$.

Let $p_r,\ q_r$ be calculated from the difference equations

$$p_r=a_rp_{r-1}+p_{r-2} \qquad q_r=a_rq_{r-1}+q_{r-2} \qquad (r>2)$$

and suppose that $F_r=\dfrac{p_r}{q_r}$ for all values of r from 3 to k. Then F_{k+1} may be found from F_k by substituting a_k+1/a_{k+1} for a_k.

Hence $$F_{k+1}=\frac{(a_k+1/a_{k+1})p_{k-1}+p_{k-2}}{(a_k+1/a_{k+1})q_{k-1}+q_{k-2}}$$

$$=\frac{a_{k+1}(a_kp_{k-1}+p_{k-2})+p_{k-1}}{a_{k+1}(a_kq_{k-1}+q_{k-2})+q_{k-1}}=\frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}}$$

Therefore $F_r=\dfrac{p_r}{q_r}$ is true also for $r=k+1$. But it is true for $r=3$, therefore for $r=4$, and so on. Hence it is true in general.

Example 15. Evaluate $1+\dfrac{1}{3+}\dfrac{1}{2+}\dfrac{1}{2+}\dfrac{1}{1+}\dfrac{1}{1+}\dfrac{1}{2}$

and calculate its convergents.

$$p_1=1,\ q_1=1; \qquad p_2=4,\ q_2=3;$$

$$\therefore\ p_3=2p_2+p_1=9; \qquad q_3=2q_2+q_1=7;$$

and $$p_4=2p_3+p_2=22; \qquad q_4=2q_3+q_2=17;$$

and $$p_5=p_4+p_3=31; \qquad q_5=q_4+q_3=24;$$

similarly it is found that $p_6=53,\ q_6=41$; $p_7=137,\ q_7=106$.

Therefore the value of the continued fraction is $\frac{137}{106}$ and its convergents are $1, \frac{4}{3}, \frac{9}{7}, \frac{22}{17}, \frac{31}{24}, \frac{53}{41}$.

Example 16. Find the r^{th} convergent of $1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\ldots$.

$$p_r = p_{r-1} + p_{r-2} \text{ and } q_r = q_{r-1} + q_{r-2}$$

But the solution of the difference equation $u_r - u_{r-1} - u_{r-2} = 0$ is of the form $u_r = A\alpha^{r-1} + B\beta^{r-1}$ where α, β are the roots of $x^2 - x - 1 = 0$, i.e. $\frac{1}{2}(1 \pm \sqrt{5})$.

$$\therefore\ p_r = c\{\tfrac{1}{2}(1+\sqrt{5})\}^{r-1} + d\{\tfrac{1}{2}(1-\sqrt{5})\}^{r-1}$$
$$q_r = g\{\tfrac{1}{2}(1+\sqrt{5})\}^{r-1} + h\{\tfrac{1}{2}(1-\sqrt{5})\}^{r-1}$$

where c, d, g, h are constants whose values can be found from $p_1 = 1, p_2 = 2, q_1 = 1, q_2 = 1$.

Thus $\quad c + d = 1, \quad c(1+\sqrt{5}) + d(1-\sqrt{5}) = 4$

hence $\quad c\sqrt{5} = \{\frac{1}{2}(1+\sqrt{5})\}^2, \quad d\sqrt{5} = -\{\frac{1}{2}(1-\sqrt{5})\}^2$

$$\therefore\ p_r\sqrt{5} = \{\tfrac{1}{2}(1+\sqrt{5})\}^{r+1} - \{\tfrac{1}{2}(1-\sqrt{5})\}^{r+1}.$$

Similarly $\quad g + h = 1. \quad g(1+\sqrt{5}) + h(1-\sqrt{5}) = 2$

hence $\quad g\sqrt{5} = \frac{1}{2}(1+\sqrt{5}), \quad h\sqrt{5} = -\frac{1}{2}(1-\sqrt{5})$

$$\therefore\ q_r\sqrt{5} = \{\tfrac{1}{2}(1+\sqrt{5})\}^r - \{\tfrac{1}{2}(1-\sqrt{5})\}^r$$

$$\therefore\ \frac{p_r}{q_r} = \frac{(1+\sqrt{5})^{r+1} - (1-\sqrt{5})^{r+1}}{2\{(1+\sqrt{5})^r - (1-\sqrt{5})^r\}}$$

If $\frac{p_r}{q_r}$ *is the* r^{th} *convergent* $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\ldots\frac{1}{+a_r}$ *of* $a_1 + \frac{1}{a_2+}\ldots\frac{1}{+a_n}$, *then* $p_r q_{r-1} - p_{r-1} q_r = (-1)^r$.

$$u_r \equiv p_r q_{r-1} - p_{r-1} q_r$$
$$= q_{r-1}(a_r p_{r-1} + p_{r-2}) - p_{r-1}(a_r q_{r-1} + q_{r-2})$$
$$= q_{r-1} p_{r-2} - p_{r-1} q_{r-2} \equiv -u_{r-1}.$$

Similarly $\quad u_{r-1} = -u_{r-2}$, etc.

$$\therefore\ u_r = (-1)^{r-2} u_2 = (-1)^r (p_2 q_1 - p_1 q_2)$$
$$= (-1)^r\{(a_1 a_2 + 1) - a_1 a_2\} = (-1)^r.$$

(Thus in Example 15, $p_7 q_6 - p_6 q_7 = 137.41 - 106.53 = -1$.)

This result shows that p_r, q_r have no common factor and hence the convergent p_r/q_r calculated from the difference equations is a fraction in its lowest terms.

The result will also be used in Example 17 to find integral solutions of an equation linear in x, y, but a more convenient method will be given in Chapter XIX, on p. 497.

Example 17. Find integral solutions of $137x - 106y = 4$.

By using the method of Example 13, we find as in Example 15 that

$$\frac{137}{106} = 1 + \frac{1}{3+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$$

and that the value of $\frac{p_6}{q_6}$ for this fraction is $\frac{53}{41}$.

Hence $$137.41 - 53.106 = (-1)^7 = -1$$

$$\therefore\ 137(-164) - 106(-212) = 4$$

$\therefore$ the given equation is satisfied by $x = -164$, $y = -212$.

Evidently it is also satisfied by

$$x = -164 + 106t \qquad y = -212 + 137t$$

and therefore by $x = 48 + 106s \qquad y = 62 + 137s \qquad (s = t - 2)$

where s is any integer or zero.

The equation ax+by=c.

In discussing the integral non-zero solutions of this equation, it may be assumed that a, b, c are integers and that a, b have no common factor. For if h is H.C.F. of a, b, the equation can have no integral solution unless h is a factor of c, and if h is a factor of c, each side may be divided by h.

When a, b have no common factor, a solution $x = x_1$, $y = y_1$ may be found from the expression of a/b as a unit continued fraction as in Example 17. Also the more general solution $x = x_1 + bt$, $y = y_1 - at$ can then be written down. There is no other solution; for if

$$ax_2 + by_2 = c = ax_1 + by_1,$$
$$a(x_2 - x_1) = b(y_1 - y_2),$$

and since a, b have no common factor, a is a factor of $y_1 - y_2$; thus $y_1 - y_2 = at$ where t is integral, and then by substitution $x_2 - x_1 = bt$;

$$\therefore\ x_2 = x_1 + bt, \quad y_2 = y_1 - at.$$

General Continued Fractions. For the continued fraction

$$a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \dots + \frac{b_n}{a_n}$$

where $a_1, a_2, \dots, b_2, b_3, \dots$ are positive or negative integers except that a_1 may be zero, if the r^{th} convergent is calculated *without removal of common factors* and is denoted by p_r/q_r, it can be proved by the methods used on pp. 244, 245 that

$$p_r = a_r p_{r-1} + b_r p_{r-2} \qquad q_r = a_r q_{r-1} + b_r q_{r-2}$$

and that

$$p_r q_{r-1} - p_{r-1} q_r = (-1)^r b_2 b_3 \dots b_r.$$

The fraction p_r/q_r will not usually be in its lowest terms.

Example 18. If $\frac{p_r}{q_r}$ is the r^{th} convergent of $a + \frac{1}{b +} \frac{1}{a +} \frac{1}{b +} \frac{1}{a +} \dots$ prove that (i) $p_{2r} = q_{2r+1}$ (ii) $bp_{2r-1} = aq_{2r}$.

$$\frac{p_{2r}}{q_{2r}} = a + \frac{1}{b +} \frac{q_{2r-2}}{p_{2r-2}} = a + \frac{p_{2r-2}}{bp_{2r-2} + q_{2r-2}}$$

$$= \frac{(ab+1)p_{2r-2} + aq_{2r-2}}{bp_{2r-2} + q_{2r-2}}.$$

But each fraction is in its lowest terms, see p. 245. Hence

$$p_{2r} = (ab+1)p_{2r-2} + aq_{2r-2}$$

$$q_{2r} = bp_{2r-2} + q_{2r-2}.$$

Also
$$q_{2r} = bq_{2r-1} + q_{2r-2},$$

$$\therefore\ p_{2r-2} = q_{2r-1}, \quad \therefore\ p_{2r} = q_{2r+1}.$$

And
$$p_{2r} - aq_{2r} = \{(ab+1)p_{2r-2} + aq_{2r-2}\} - a(bp_{2r-2} + q_{2r-2})$$

$$= p_{2r-2}.$$

But
$$p_{2r} = bp_{2r-1} + p_{2r-2},$$

$$\therefore\ bp_{2r-1} = aq_{2r}.$$

EXERCISE XIc

A

1. Express $\frac{29}{11}$ as a unit continued fraction.

2. Verify that $\sqrt{5} = 2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{x}$ and find the value of x.

3. A strip of paper is divided into 48 equal lengths by red lines and into 35 equal lengths by blue lines. Which pair of red and blue lines are closest together?

4. If p_r/q_r is the r^{th} convergent of $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\dots$ prove that $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n$.

5. Prove that the r^{th} convergent of $2 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2}\dots$ is

$$\{(1+\sqrt{2})^{r+1} - (1-\sqrt{2})^{r+1}\} \div \{(1+\sqrt{2})^r - (1-\sqrt{2})^r\}.$$

6. If p_r/q_r is the r^{th} convergent of $a + \frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\dots$ prove that $p_{n+2} - (ab+2)p_n + p_{n-2} = 0$ and that p_n is the coefficient of x^n in the expansion of $(1 + ax - x^2) \div \{1 - (ab+2)x^2 + x^4\}$

7. Solve in integers: $11x - 9y = 4$.

8. Find the least positive integral solution of $19x - 117y = 11$.

9. Find the number of solutions in positive integers of $7x + 9y = 1000$.

10. If p_r/q_r is the r^{th} convergent of $a + \frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\dots$ prove that $p_{n-1} = q_n$ and $p_n = aq_n + q_{n-1}$.

Find the r^{th} convergents of the continued fractions in Nos. 11, 12.

11. $\frac{6}{1+}\frac{6}{1+}\frac{6}{1+}\dots$

12. $1 - \frac{1}{2-}\frac{1}{2-}\frac{1}{2-}\dots$

B

13. Express $\frac{41}{25}$ as a unit continued fraction.

14. Verify that $\sqrt{11} = 3 + \frac{1}{3+}\frac{1}{6+}\frac{1}{3+}\frac{1}{6+}\frac{1}{3+}\frac{1}{x}$ and find the vaue of x.

15. If p_r/q_r is the r^{th} convergent of $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\dots$ prove tlat $(p_{n+1} - p_{n-1})/p_n = (q_{n+1} - q_{n-1})/q_n$.

16. Prove that the r^{th} convergent of $1+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\dots$ is $\frac{1}{2}\{(1+\sqrt{3})^{r+1}-(1-\sqrt{3})^{r+1}\}\div\{(1+\sqrt{3})^r-(1-\sqrt{3})^r\}$

17. Solve in integers : $13x+17y=8$.

18. Find the number of solutions in positive integers of $11x+15y=784$.

Find the r^{th} convergents of the continued fractions in Nos. 19, 20.

19. $1-\frac{2}{3-}\frac{2}{3-}\frac{2}{3-}\dots$

20. $\frac{1}{1-}\frac{1}{4-}\frac{1}{1-}\frac{1}{4-}\frac{1}{1-}\dots$

C

21. Verify that $\sqrt{19}=4+\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{x}$ and find the value of x.

22. Express 3·1416 as a unit continued fraction and find the first three convergents.

23. Prove that the r^{th} convergent of $2a+\frac{1}{a+}\frac{1}{4a+}\frac{1}{a+}\frac{1}{4a+}\dots$ is double that of $a+\frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\dots$

24. If p_r/q_r is the r^{th} convergent of $a+\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\dots$ prove that $p_{3n}q_3-p_3q_{3n}=q_{3n-3}$.

25. Solve $4x+7y+19z=69$ in positive integers.

26. Prove that the number of positive integral or zero solutions of $x+2y=n$ where n is a positive integer is $\frac{1}{4}\{2n+3+(-1)^n\}$.

27. Prove the results stated on p. 247 for the continued fraction $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots$

Find the r^{th} convergents of the continued fractions in Nos. 28, 29.

28. $\frac{1}{1+}\frac{1}{2+}\frac{3}{2+}\frac{3}{2+}\frac{3}{2+}\dots$

29. $\frac{a}{(a-1)+}\frac{a}{(a-1)+}\frac{a}{(a-1)+}\dots$

30. Prove that the $3n^{\text{th}}$ convergent of

$$\frac{1}{5-}\frac{1}{2-}\frac{1}{1-}\frac{1}{5-}\frac{1}{2-}\frac{1}{1-}\frac{1}{5-}\dots \text{ is } \frac{n}{3n+1}$$

31. If $x = a + \frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\dots$ and $y = b + \frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\dots$ prove that $bx = ay$.

32. If p_r/q_r is the r^{th} convergent of $a_1 + \frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots$ prove that

$$\text{(i)}\ \frac{p_n}{p_{n-1}} = a_n + \frac{b_n}{a_{n-1}+}\dots+\frac{b_2}{a_1}$$

$$\text{(ii)}\ \frac{q_n}{q_{n-1}} = a_n + \frac{b_n}{a_{n-1}+}\dots+\frac{b_3}{a_2}$$

33. Prove that $\frac{2}{2-}\frac{3}{3-}\frac{4}{4-}\dots\frac{n+2}{-n+2} = 1 + \sum_1^n r!$

MISCELLANEOUS EXAMPLES

EXERCISE XId

A

1. Find u_r if $u_0 = 1$, $u_1 = 3$ and $u_r - 9u_{r-1} + 20u_{r-2} = 0$, $r \geqslant 2$.

2. Find the r^{th} term and the sum to n terms of the recurring series $-1, -1, 7, 71, \dots$ of order 2.

3. Find u_r if $u_1 = b$ and $u_r - au_{r-1} = b$, $r \geqslant 2$

(i) if $a \neq 1$, (ii) if $a = 1$.

4. Find the generating function and the coefficient of x^r for the recurring power series $4 + x + 7x^2 - 5x^3 + \dots$, order 2.

5. If $u_1 = 3$, $u_2 = 12$ and if $u_r - (2r+1)u_{r-1} + (r^2 - 1)u_{r-2} = 0$, $r \geqslant 3$, find u_r.

6. If $2u_{r+1} = u_r + a^2/u_r$, prove that

$$(u_r - a)/(u_r + a) = \{(u_1 - a)/(u_1 + a)\}^p \text{ where } p = 2^{r-1}.$$

7. If $u_r = \phi(r)$ is any solution of $u_r + a_1 u_{r-1} + a_2 u_{r-2} = f(r)$ where a_1, a_2 are independent of r, and if $u_r = g(r)$ and $u_r = h(r)$ are any two independent solutions of $u_r + a_1 u_{r-1} + a_2 u_{r-2} = 0$, prove that the general solution of the first difference equation is $u_r = \phi(r) + Bg(r) + Ch(r)$ where B and C are arbitrary constants. Hence solve the difference equations (i) $u_r - 5u_{r-1} + 6u_{r-2} = a$, (ii) $u_r - 5u_{r-1} + 6u_{r-2} = ar$.

8. Prove that the n^{th} convergent of $x - \frac{1}{x-}\frac{1}{x-}\frac{1}{x-}\dots$ where $x = 2\cos\theta$, is $\sin(n+1)\theta \operatorname{cosec} n\theta$.

9. If p_r/q_r and P_r/Q_r are the r^{th} convergents of $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\dots$ and $a_2 + \frac{1}{a_3+}\frac{1}{a_4+}\dots$ prove that (i) $q_n = P_{n-1}$ (ii) $p_n = a_1 q_n + Q_{n-1}$

10. If $n = 6N + r$ where N and r are positive integers, prove that the number of positive integral or zero solutions of $2x + 3y = n$ is $N + 1$ unless $r = 1$ when it is N.

B

11. Find the r^{th} term and the sum to n terms of the recurring series $-1, 1, 6, 19, 54, 151, \dots$ of order 3.

Find the generating function and the coefficient of x^r for the recurring power series in Nos. 12, 13.

12. $1 + 4x + 15x^2 + 54x^3 + \dots$ of order 2.

13. $1 - 4x^2 - 18x^3 - 64x^4 - 210x^5 - 664x^6 - \dots$ of order 3.

14. Find u_r if $u_0 = 0$, $u_1 = 1$, and $u_r + u_{r-1} + u_{r-2} = 0$ for $r \geqslant 2$.

15. If $u_r = u_{r-1} + u_{r-2}$, $r \geqslant 3$, prove that $u_r = 3u_{r-2} - u_{r-4}$ and that if $u_1 = u_2 = 1$, $u_r^2 - u_{r+2}u_{r-2} = (-1)^r$.

16. If p_n/q_n is the n^{th} convergent of $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\dots$ prove that $p_n q_{n-4} - p_{n-4} q_n = (-1)^{n-1}(a_n a_{n-1} a_{n-2} + a_n + a_{n-2})$

17. If p_r/q_r is the r^{th} convergent of $1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\dots$ prove that $3q_{3n} = 2p_{3n} + p_{3n-3}$.

18. If p_r/q_r is the r^{th} convergent of $\frac{a}{b+}\frac{c}{d+}\frac{a}{b+}\frac{c}{d+}\dots$ prove that (i) $bp_{2n} = adq_{2n-1}$ (ii) $p_n - (a + c + bd)p_{n-2} + acp_{n-4} = 0$.

C

19. Find u_r if $u_r - (2r+1)u_{r-1} + r^2 u_{r-2} = 0$ for $r \geqslant 2$, and if (i) $u_0 = 1$, $u_1 = 2$, (ii) $u_0 = 1$, $u_1 = 3$.

20. Find the generating function of the recurring power series

$$(a - b) + 2(a^2 - b^2)x + 3(a^3 - b^3)x^2 + 4(a^4 - b^4)x^3 + \dots$$

21. If $u_r - u_{r-1} - u_{r-2} + u_{r-3} = a$, $r \geqslant 4$, express the value of $\sum_1^{2n} u_r$ in terms of u_1, u_2, u_3, a.

22. If $u_r = \alpha^r + \beta^r + \gamma^r$ and if $u_1 = a$, $u_2 = b^2$, $u_3 = c^3$, prove that the generating function of the recurring power series $u_0 + u_1 x + u_2 x^2 + \dots$, is

$\{3 - 2ax + \frac{1}{2}(a^2 - b^2)x^2\} \div \{1 - ax + \frac{1}{2}(a^2 - b^2)x^2 - \frac{1}{6}(a^3 - 3ab^2 + 2c^3)x^3\}$

and express u_r in terms of u_{r-1}, u_{r-2}, u_{r-3}, a, b, c.

Find the general solutions of the difference equations in Nos. 23, 24.

23. $u_r - a^2 u_{r-2} = \cos br$ **24.** $u_r - 4u_{r-1} + 4u_{r-2} = 2^r$

25. If p_r/q_r and P_r/Q_r are the r^{th} convergents of $a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +}\ldots$ and $a_n + \frac{1}{a_{n-1} +}\frac{1}{a_{n-2} +}\ldots$ prove that

$$p_n = p_{n-r}P_r + p_{n-r-1}P_{r-1} = P_n \text{ for } 1 < r < n-1.$$

26. Prove that $\frac{1}{2-}\frac{2}{3-}\frac{3}{4-}\ldots\frac{n}{n+1} = \left(\sum_1^n r!\right) \div \left(1 + \sum_1^n r!\right)$

27. If a rational number F is expressed as a unit continued fraction and if p_r/q_r is its r^{th} convergent, prove that

(i) $|F - p_r/q_r| < |F - p_{r-1}/q_{r-1}|$

(ii) $1/(q_r q_{r+1}) > |F - p_r/q_r| > 1/\{q_r(q_r + q_{r+1})\}$

28. With the notation of No. 27, prove that

$$\frac{p_r p_{r+1}}{q_r q_{r+1}} \gtrless F^2 \text{ according as } \frac{p_r}{q_r} \gtrless \frac{p_{r+1}}{q_{r+1}}$$

29. Prove that $u_1 - u_2 + u_3 - \ldots + (-1)^{n-1}u_n$

$$= \frac{u_1}{1+}\frac{u_2}{(u_1 - u_2)+}\frac{u_1 u_3}{(u_2 - u_3)+}\ldots\frac{u_{n-2}u_n}{(u_{n-1} - u_n)}$$

Hence express the sum to n terms of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ as a continued fraction.

30. n equal uniform rods A_0A_1, A_1A_2, ..., $A_{n-1}A_n$, each of mass m, are freely jointed at A_1, A_2, ..., A_{n-1} and rest in a straight line on a horizontal table. A horizontal impulse I_0 is applied at A_0 perpendicular to A_0A_1. If the instantaneous impulse at the joint A_r is I_r, it can be proved that $I_r - 4I_{r-1} + I_{r-2} = 0$, $2 < r < n$ where $I_n = 0$. Prove that $I_r = I_0 \operatorname{sh}(n-r)\alpha / \operatorname{sh} n\alpha$ where $\alpha = \log(2 + \sqrt{3})$.

CHAPTER XII

FACTORS AND PARTIAL FRACTIONS

Complex Algebra. There are certain phrases in common use which are so misleading to the student that they ought to be avoided.

The ordered pairs $[p, q]$ of real numbers which occur in complex algebra are usually written in the form $p+qi$ and they are called complex numbers.

It has become customary to call $p+qi$ a real number when $q=0$ and a pure imaginary number when $p=0$. This confuses a pair of numbers one of which is zero with a single number. Also complex numbers in general are sometimes called imaginary. This use of the words 'real' and 'imaginary' dates from a time when the theory of complex numbers was imperfectly understood. It is perhaps too much to hope that the old form of words will soon be discarded; but it seems essential in dealing with certain fundamental parts of the subject to use words that are not open to misinterpretation. Accordingly we shall call the number $p+qi$ an **x-axal** number when $q=0$, a **y-axal** number when $p=0$, and a **non-axal** number when $p \neq 0$ and $q \neq 0$. In the last sentence the word 'number' was used as an abbreviation for 'complex number'. This is customary in complex algebra.

The system of real numbers is one-dimensional, that of complex algebra is two-dimensional. The importance of x-axal numbers in complex algebra is due to the exact correspondence existing between x-axal numbers and real numbers, and between their properties. This correspondence makes possible the deduction from any theorem about x-axal numbers of a corresponding theorem in real algebra. Complex algebra is in fact often of use to a person who is investigating properties of real numbers. For example the method used at the present day to investigate the distribution of prime numbers depends on the theory of functions of a complex variable.

Factorisation. It is fundamental in the theory of factorisation that in complex algebra a polynomial of degree n can be written as the product of n linear factors. In real algebra this is not necessarily true, but a polynomial is the product of factors which may be some linear and the others quadratic.

The proof of these statements depends on D'Alembert's or Gauss' theorem :

In complex algebra, the equation

$$f(z) \equiv z^n + a_1 z^{n-1} + \ldots + a_n = 0 \qquad (1)$$

has at least one root.

An elementary proof which itself however depends on a fundamental theorem in analysis on bounds will be found in *Burnside and Panton's Theory of Equations, Vol. I, p. 260 ;* but a simpler proof (*id., p. 258*) can be obtained by using Cauchy's theorem on contour integration (*Hardy : Pure Mathematics, Appendix I ; Whittaker and Watson : Modern Analysis, p. 120*) ; the student is therefore advised to postpone reading any proof of this theorem until he is in a position to appreciate Cauchy's method.

Multiple Roots. The equation $(z-\alpha)^r g(z)=0$ where $g(z)$ is a polynomial in z, $g(\alpha) \neq 0$, and $r=2$ or 3 or 4 or ... , is said to have an r-fold root $z=\alpha$, and it is also convenient to say that it has r roots $z=\alpha$.

In particular if $r=2$, $z=\alpha$ is called a *double* root.

Roots of the General Equation in Complex Algebra.

By using D'Alembert's theorem and the above convention about multiple roots it is easy to prove that

in complex algebra the equation

$$f(z) \equiv z^n + a_1 z^{n-1} + \ldots + a_n = 0$$

has exactly n roots, any r-fold root being reckoned r times.

By D'Alembert's theorem, there is a root $z=\alpha_1$

$$\therefore \quad \alpha_1{}^n + a_1\alpha_1{}^{n-1} + \ldots + a_n = 0.$$

Hence the equation is equivalent to

$$(z^n - \alpha_1{}^n) + a_1(z^{n-1} - \alpha_1{}^{n-1}) + \ldots + a_{n-1}(z - \alpha_1) = 0$$

and therefore to $\qquad (z-\alpha_1)f_1(z) = 0$

where $f_1(z)$ is a polynomial of degree $n-1$ in z.

Similarly since $f_1(z)=0$ has a root $z=\alpha_2$

$$f_1(z) \equiv (z-\alpha_2)f_2(z).$$

By continuing this process it follows that

$$f(z) \equiv (z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1})f_{n-1}(z)$$

where $f_{n-1}(z)$ is linear and must be of the form $z-\alpha_n$ because the coefficient of z^n in $f(z)$ is unity. Hence

$$f(z) \equiv (z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

or using the notation for products,

$$f(z) \equiv \prod_{r=1}^{n}(z-\alpha_r).$$

Therefore the equation $f(z)=0$ has the n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ which need not however be distinct.

It also follows from (2) that $f(z)$ is not zero for any value of z except $\alpha_1, \alpha_2, \ldots, \alpha_n$. Therefore the number of roots is exactly n, subject to the convention about multiple roots.

Thus the polynomial $z^n + a_1z^{n-1} + \ldots + a_n$ cannot be zero for more than n values of z, and the same is true for the polynomial $a_0z^n + a_1z^{n-1} + \ldots + a_n$ where $a_0 \neq 0$, since it can be written in the form

$$a_0(z^n + b_1z^{n-1} + \ldots + b_n) \equiv a_0(z-\beta_1)\ldots(z-\beta_n).$$

But if $a_0=0$, it reduces to $a_1z^{n-1} + \ldots + a_n$, and similarly this cannot be zero for n values of z if $a_1 \neq 0$, and so on, thus

if a polynomial $a_0z^n + a_1z^{n-1} + \ldots + a_n$ is zero for more than n values of z, each coefficient is zero ;

also, *if the polynomials*

$$b_0z^n + b_1z^{n-1} + \ldots + b_n \text{ and } c_0z^n + c_1z^{n-1} + \ldots + c_n$$

are equal for more than n values of z, then $b_r = c_r$ for $r = 0, 1, 2, \ldots, n$.

If the roots of (1) are not all distinct, (2) takes the form

$$f(z) \equiv (z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2} \dots (z-\alpha_k)^{n_k} \equiv \prod_{r=1}^{k} (z-\alpha_r)^{n_r} \quad \dots\dots(3)$$

where $n_1 + n_2 + \dots + n_k = n$, and this form is *unique*. For if also

$$f(z) \equiv \prod_{r=1}^{h} (z-\beta_r)^{m_r} \quad \dots\dots\dots\dots\dots\dots(4)$$

then each β must be an α because otherwise from (3) $f(\beta_r) \neq 0$ and from (4) $f(\beta_r) = 0$. Similarly each α must be a β. Thus (4) can be written $f(z) \equiv \prod_{r=1}^{k} (z-\alpha_r)^{m_r}$ and here m_r must equal n_r, for otherwise by equating this form of $f(z)$ to the form in (3) and simplifying we obtain an identity of the form

$$\Pi(z-\gamma_s)^{p_s} \equiv \Pi(z-\delta_t)^{p_t}$$

where p_s, p_t are positive integers and each γ is different from each δ, and this has just been proved to be impossible.

Example 1. Solve the simultaneous equations

$$f(\lambda_\nu) \equiv \frac{z_1}{a_1+\lambda_\nu} + \frac{z_2}{a_2+\lambda_\nu} + \dots + \frac{z_n}{a_n+\lambda_\nu} - 1 = 0, \ (\nu = 1 \text{ to } n),$$

for $z_1, z_2, \dots, z_n$.

These equations imply that

$$(a_1+\lambda)(a_2+\lambda)\dots(a_n+\lambda)\left\{\frac{z_1}{a_1+\lambda} + \frac{z_2}{a_2+\lambda} + \dots + \frac{z_n}{a_n+\lambda} - 1\right\}$$

is zero for $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$. And the expanded value of this expression is a polynomial of degree n in λ in which the leading term is $-\lambda^n$. Hence it is identical with

$$-(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n).$$

Putting $\lambda = -a_k$ in the identity,

$$(a_1-a_k)(a_2-a_k)\dots(a_{k-1}-a_k)(a_{k+1}-a_k)\dots(a_n-a_k)z_k$$
$$= (-1)^{n+1}(a_k+\lambda_1)(a_k+\lambda_2)\dots(a_k+\lambda_n).$$

This gives z_k for $k = 1, 2, \dots, n$.

Conjugate Complex Roots. If $\alpha+\beta i$, where $\beta\neq 0$, is a root of $f(z)=0$ where $f(z)$ is a polynomial with x-axal coefficients, then the *conjugate complex* $\alpha-\beta i$ is also a root.

The conditions for $\alpha+\beta i$, $\alpha-\beta i$ to be roots are of the forms $P+Qi=0$, $P-Qi=0$. These are both the same as $P=Q=0$ by the elementary theory of complex numbers.

Also if $\alpha+\beta i$ is an r-fold root of $f(z)=0$ so that

$$f(z)\equiv(z-\alpha-\beta i)^r g(z)$$

where $g(z)$ is a polynomial, it follows by the same principle that

$$f(z)\equiv(z-\alpha+\beta i)^r h(z)$$

where $h(z)$ is another polynomial.

Hence $\alpha-\beta i$ is an s-fold root, where $s\geqslant r$; similarly $r\geqslant s$, so $s=r$.

Since $(z-\alpha-\beta i)(z-\alpha+\beta i)=(z-\alpha)^2+\beta^2$, any polynomial $f(z)$ of degree n with x-axal coefficients can be expressed uniquely in the form

$$f(z)\equiv a_0(z-p_1)\ldots(z-p_k)\{(z-q_1)^2+r_1^2\}\ldots\{(z-q_l)^2+r_l^2\}\ldots(5)$$

where $k+2l=n$ and all constants are x-axal. The factors are not necessarily distinct and in special cases it will happen that all are linear or all quadratic.

Roots of the General Equation in Real Algebra.

If $f(x)\equiv a_0x^n+a_1x^{n-1}+\ldots+a_n=0$, where $a_0\neq 0$, is the general equation of degree n in real algebra, there is an identity

$$f(x)\equiv a_0(x-p_1)\ldots(x-p_k)\{(x-q_1)^2+r_1^2\}\ldots\{(x-q_l)^2+r_l^2\}\ldots(6)$$

where $k+2l=n$, corresponding to the identity (5). This is due to the exact correspondence between real numbers and x-axal complex numbers. If the identity (5) is verified by multiplication of the factors, precisely the same work (with x instead of z) will verify the identity (6).

Since $k+2l=n$, there must be at least one factor of the form $x-p$ if n is odd; but the factors may all be quadratic if n is even, and no quadratic factor of the form $(x-q)^2+r^2$ can be zero.

Hence we have the following theorem:

in real algebra an equation of even degree has either no roots or an even number of roots, and an equation of odd degree has an odd number of roots.

It follows that an equation of odd degree has at least one root.

Binomial Equations. The equation $z^n = 1$ may be solved by De Moivre's theorem. Writing cis θ for $\cos\theta + i\sin\theta$,

$$z^n = \text{cis } 2k\pi, \quad \therefore\ z = \text{cis } (2k\pi/n),\ k = 0, 1, 2, \dots, n-1.$$

In particular the roots of $z^3 = 1$ are 1, cis $\frac{2}{3}\pi$, cis $\frac{4}{3}\pi$ or 1, ω, ω^2 where $\omega = \text{cis } \frac{2}{3}\pi$, i.e. 1, $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$. These are the *cube roots of unity*, and $\omega^2 + \omega + 1 = 0$.

Example 2. Factorise $x^3 + y^3 + z^3 - 3xyz$.

$$\text{The expression} = \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \begin{vmatrix} x+y+z & y & z \\ z+x+y & x & y \\ y+z+x & z & x \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix} = (\Sigma x)(\Sigma x^2 - \Sigma yz)$$

But the expression can be written in the form,

$$x^3 + (\omega y)^3 + (\omega^2 z)^3 - 3x(\omega y)(\omega^2 z),$$

$\therefore\ x + \omega y + \omega^2 z$ is a factor, and similarly $x + \omega^2 y + \omega z$ is a factor. But the coefficient of x^3 in the expression is unity,

$$\therefore\ x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

EXERCISE XIIa

A

1. State the number of roots of (i) $x^4 = 1$ in real algebra, (ii) $z^4 = 1$ in complex algebra.

2. Prove that $1 + \omega^4 + \omega^8 = 0$

3. Prove that

$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \equiv a^2 + b^2 + c^2 - bc - ca - ab.$$

4. Express $x^3 + y^3$ as the product of three factors.

5. Solve the equation $z^4 - 3z^2 - 6z - 2 = 0$, given that $-1 + i$ is a root.

6. If $a + bi = (x + yi)^n$ where a, b, x, y are x-axal, express $a^2 + b^2$ in terms of x, y.

7. If the coefficients of the equation $z^4 + pz^3 + qz^2 + rz + s = 0$ are x-axal and there is a y-axal root, prove that $r^2 + p^2 s = pqr$.

8. If n is a positive integer not divisible by 3, prove that $x^{2n} + 1 + (x+1)^{2n}$ is divisible by $x^2 + x + 1$.

9. Find the fifth roots of unity algebraically by using the method of p. 163, No. 19. If two of them are 1 and ϵ, express the others in terms of ϵ.

10. Solve for x, y, z the equations

$$x + ay + a^2 z = a^3 \qquad x + by + b^2 z = b^3 \qquad x + cy + c^2 z = c^3.$$

B

11. Solve $z^3 + 1 \equiv (z+1)(z^2 - z + 1) = 0$ in complex algebra.

12. Prove that $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$.

13. Express $x^3 + y^3 - z^3 + 3xyz$ as the product of three factors.

14. Solve the equation $2z^4 - z^3 - 3z^2 - 5z - 2 = 0$, given that ω is a root.

15. If $a + bi = (c + di)^2$ where a, b, c, d are x-axal, form in terms of c, d the equations whose roots are (i) $a \pm bi$, (ii) $b \pm ai$.

16. Prove that the roots of $z^n = 1$ are all distinct.

C

17. Prove that

$$(a + \omega b + \omega^2 c)^3 - (a + \omega^2 b + \omega c)^3 \equiv -3i\sqrt{3}(b-c)(c-a)(a-b).$$

18. Find the condition that one root of the equation

$$z^2 + 2(a + bi)z + c + di = 0$$

is x-axal if a, b, c, d are x-axal.

19. For what positive integral values of n is

$$(y-z)^n + (z-x)^n + (x-y)^n$$

divisible by $x^2 + y^2 + z^2 - yz - zx - xy$?

20. Solve for x, y, z the equations

$$x + ay + a^2 z = a^4 \qquad x + by + b^2 z = b^4 \qquad x + cy + c^2 z = c^4.$$

21. If $\lambda + \mu i$ is a root of $z^4 + 4bz + c = 0$ where b and c are x-axal, prove that $\mu^2 = \lambda^2 + b/\lambda$ and that $2\lambda^2$ is a root of $z^3 - cz - 2b^2 = 0$.

22. If $z^n + a_1 z^{n-1} + \ldots + a_n \equiv (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$, prove that $(1 + \alpha_1^2)(1 + \alpha_2^2) \ldots (1 + \alpha_n^2) \equiv (1 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2$.

23. If $(1 + x + x^2)^n \equiv a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, prove that

$$a_0 + a_3 + a_6 + \ldots = a_1 + a_4 + a_7 + \ldots = a_2 + a_5 + a_8 + \ldots$$

24. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation

$$f(z) \equiv z^n + a_1 z^{n-1} + \ldots + a_n = 0,$$

prove that

$$\sum_1^n \frac{f(z)}{z - \alpha_r} \equiv nz^{n-1} + (n-1)a_1 z^{n-2} + (n-2)a_2 z^{n-3} + \ldots + a_{n-1}$$

Highest Common Factor. If polynomials $f(x)$, $g(x)$ have a common factor $h(x)$, this will also be a factor of $Af(x) + Bg(x)$ where A, B are any two polynomials or constants. If there is no common factor of higher degree than $h(x)$, then $h(x)$ is called the highest common factor (H.C.F.) of $f(x)$, $g(x)$. Numerical factors are regarded as irrelevant so that $Ch(x)$ where C is any constant other than zero is also called the H.C.F.

The process used in arithmetic of finding an H.C.F. by repeated division may be applied to polynomials.

Suppose that the degree n of $f(x)$ is not less than that of $g(x)$ and let $r(x)$ be the remainder when $f(x)$ is divided by $g(x)$. Then the degree of $r(x)$ is not greater than $n - 1$, and

$$r(x) \equiv f(x) + Ag(x)$$

where A is a polynomial or constant. Thus any common factor of $f(x)$, $g(x)$ is also a factor of $r(x)$, and any common factor of $r(x)$, $g(x)$ is also a factor of $f(x)$. Therefore the H.C.F. of $f(x)$, $g(x)$ is the same as that of $g(x)$, $r(x)$.

Similarly if $r_1(x)$ is the remainder when $g(x)$ is divided by $r(x)$, the H.C.F. of $g(x)$, $r(x)$ is the same as that of $r(x)$, $r_1(x)$ and the degree of $r_1(x)$ is not greater than $n - 2$, and so on. In this way after n divisions at most, a remainder r_k is found which is a constant, possibly zero.

If $r_k = 0$, $r_{k-1}(x)$ is a factor of $r_{k-2}(x)$ and is the H.C.F. of $f(x)$, $g(x)$.

If $r_k \neq 0$, $f(x)$ and $g(x)$ have no common factor since such a factor must also be a factor of r_k.

Polynomials which have no common factor are called *co-prime.*

Example 3. Find the H.C.F. of $x^5 - 3x^3 + 5x^2 + 11x - 4$ and $2x^4 - 5x^3 + 3x^2 + 10x - 8$.

It is convenient to begin by dividing $2(x^5 - 3x^3 + 5x^2 + 11x - 4)$ by $2x^4 - 5x^3 + 3x^2 + 10x - 8$ and to use detached coefficients.

(×7)	2 − 5 + 3 + 10 − 8	2 + 0 − 6 + 10 + 22 − 8	
	14 − 35 + 21 + 70 − 56	2 − 5 + 3 + 10 − 8	
	14 − 30 + 20 + 48	5 − 9 + 0 + 30 − 8	(×2)
(×7)	− 5 + 1 + 22 − 56	10 − 18 + 0 + 60 − 16	
	− 35 + 7 + 154 − 392	10 − 25 + 15 + 50 − 40	
	− 35 + 75 − 50 − 120	7 − 15 + 10 + 24	
(÷ − 68)	− 68 + 204 − 272	7 − 21 + 28	
	1 − 3 + 4	6 − 18 + 24	
		6 − 18 + 24	

Thus the H.C.F. is $x^2 - 3x + 4$.

The introduction of the numerical factors (to avoid fractions) is justified by the fact that the remainders are still of the form $Af(x) + Bg(x)$. Another useful device is to make the constant term in the remainder zero. The factor x can then be removed, since it will be obvious whether it should be included in the H.C.F. Thus the working of Example 3 might be arranged:

(×3)	2 − 5 + 3 + 10 − 8	2 + 0 − 6 + 10 + 22 − 8
	6 − 15 + 9 + 30 − 24	2 − 5 + 3 + 10 − 8
	6 − 8 − 6 + 40	5 − 9 + 0 + 30 − 8
(×3)	− 7 + 15 − 10 − 24	2 − 5 + 3 + 10 − 8
	− 21 + 45 − 30 − 72	3 − 4 − 3 + 20
	21 − 28 − 21 + 140	3 − 9 + 12
(÷17)	17 − 51 + 68	5 − 15 + 20
	1 − 3 + 4	5 − 15 + 20

If at any stage a remainder can be factorised by inspection, it can be found by trial which factors belong to the H.C.F. In this example $3x^3 - 4x^2 - 3x + 20$ could be factorised.

Two theorems required in the theory of equations are added here.

If $f(x)$, $g(x)$ are polynomials with rational coefficients which are both zero for $x=a$, and if $g(x)$ cannot be expressed as a product of two polynomials with rational coefficients, then $g(x)$ is a factor of $f(x)$.

Since $f(a)=0=g(a)$, $x-a$ is a common factor of $f(x)$, $g(x)$. Hence the H.C.F. exists. But the process of repeated division shows that the coefficients of the H.C.F. must be rational. As $g(x)$ has no factor with rational coefficients, the H.C.F. must be $g(x)$.

Example 4. If $1+\sqrt{2}+\sqrt{3}$ is a root of the equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+\ldots+a_n=0$$

in which the coefficients are rational, prove that $1\pm\sqrt{2}\pm\sqrt{3}$ are also roots.

The function

$$(x-1-\sqrt{2}-\sqrt{3})(x-1+\sqrt{2}+\sqrt{3})(x-1-\sqrt{2}+\sqrt{3})(x-1+\sqrt{2}-\sqrt{3})$$
$$\equiv\{(x-1)^2-5\}^2-24\equiv g(x)$$

has rational coefficients and has no two factors with rational coefficients. Also $f(x)$, $g(x)$ are zero for $x=1+\sqrt{2}+\sqrt{3}$. Hence $g(x)$ is a factor of $f(x)$. Hence $1\pm\sqrt{2}\pm\sqrt{3}$ are roots of $f(x)=0$.

Repeated Factors. *If $(x-a)^p$ is a factor of the polynomial $f(x)$, then $(x-a)^{p-1}$ is a factor of its derivative.*

Also if $(x-a)^p$ is a common factor of $f(x)$ and $f'(x)$, then $(x-a)^{p+1}$ is a factor of $f(x)$.

$$f(x)\equiv(x-a)^p g(x) \text{ where } g(x) \text{ is a polynomial,}$$

$$\therefore\ f'(x)\equiv(x-a)^p g'(x)+p(x-a)^{p-1}g(x),$$

$$\therefore\ (x-a)^{p-1} \text{ is a factor of } f'(x).$$

If also $(x-a)^p$ is a factor of $f'(x)$, the last identity shows that it is also a factor of $p(x-a)^{p-1}g(x)$. Therefore $x-a$ is a factor of $g(x)$. Hence $(x-a)^{p+1}$ is a factor of $f(x)$.

A similar argument shows that if $(x^2+2bx+c)^p$ where $b^2 \neq c$, is a common factor of $f(x)$ and $f'(x)$, then $(x^2+2bx+c)^{p+1}$ is a factor of $f(x)$.

In virtue of these results the H.C.F. process can be used to find any repeated factors of a polynomial or any multiple roots that may exist of a given algebraic equation.

Example 5. Solve the equation

$$f(x) \equiv x^6 - 16x^4 + 20x^3 + 13x^2 - 4x - 2 = 0$$

given that there are equal roots.

$$f'(x) \equiv 2(3x^5 - 32x^3 + 30x^2 + 13x - 2).$$

The H.C.F. process shows that $x^2 - 2x - 1$ is a common factor of $f(x)$, $f'(x)$. Hence $(x^2-2x-1)^2$ is a factor of $f(x)$.

Actually $\qquad f(x) \equiv (x^2-2x-1)^2(x^2+4x-2)$

$\therefore$ the roots are $1 \pm \sqrt{2}$ repeated and $-2 \pm \sqrt{6}$.

EXERCISE XIIb

A

Find the H.C.F. of the pairs of polynomials in Nos. 1, 2.

1. $2x^3 - 3x^2 - 5x + 6$, $3x^3 - 8x^2 + 3x + 2$.

2. $4x^4 - 8x^3 - 3x^2 + 7x - 2$, $2x^3 - 9x^2 + 12x - 4$.

3. Factorise $4x^4 - 35x^2 + 51x - 18$, given that it has a repeated factor.

4. Given that $1 + \sqrt{2}$ is a root of $x^4 - x^3 - 4x^2 + x + 1 = 0$, find the other roots.

5. Find the condition for $x^3 - 3Hx + G = 0$ to have two equal roots.

6. Prove that $(y-z)^n + (z-x)^n + (x-y)^n$ is divisible by $(x^2+y^2+z^2-yz-zx-xy)^2$ if $n-1$ is a positive multiple of 3.

7. Prove that if $n > 1$, $(x-1)^3$ is a factor of

$$2x^{n+2} - (n+1)(n+2)x^2 + 2n(n+2)x - n(n+1)$$

B

Find the H.C.F. of the pairs of polynomials in Nos. 8-11.

8. $x^3 - 2x^2 - 5x - 12$, $x^3 - 7x^2 + 13x - 4$

9. $2x^3 + 9x^2 + 4x - 15$, $4x^3 + 8x^2 + 3x + 20$

10. $2x^4 + x^3 + 2x^2 + 1$, $x^3 + 2x^2 + 2x + 1$

11. $3x^4 + 5x^3 + x^2 - 10x - 14$, $3x^3 + 14x^2 + 22x + 21$

12. Given that $\sqrt{2} + \sqrt{5}$ is a root of

$$x^6 + 2x^5 - 15x^4 - 28x^3 + 23x^2 + 18x = 9,$$

find all the roots.

13. Solve $4x^4 + 4x^3 - 23x^2 + 18x = 4$, given that the equation has two equal roots.

14. For what positive integral values of n is $x^{2n} + 1 + (x+1)^{2n}$ divisible by $(x^2 + x + 1)^2$?

15. Prove that the equation

$$x^n + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n! = 0$$

has no equal roots.

C

16. Find the condition that the H.C.F. of $x^3 + ax^2 + bx + c$ and $x^3 + bx^2 + ax + c$ is not a mere constant.

17. Find the condition for $x^4 + 2px^3 + 2rx + pr = 0$ to have two equal roots.

18. Explain the necessity for the condition $b^2 \neq c$ in the statement on p. 263.

19. If $z^4 - 4cz + 3 = 0$ has two equal roots in complex algebra, find c and solve the equation.

20. If $f(x)$, $g(x)$ are co-prime polynomials in x such that $f^2 + g^2 = 0$ has a repeated root, prove that this root satisfies $f'^2 + g'^2 = 0$ where f', g' are the derivatives of f, g.

21. If $f(x, y)$ is a symmetrical polynomial in x and y having $x - y$ as a factor, prove that $(x-y)^2$ is a factor of $f(x, y)$.

H.C.F. Theorems. *If $h(x)$ is the H.C.F. of two polynomials $f(x)$, $g(x)$, then co-prime polynomials A, B exist such that*

$$Af(x)+Bg(x)\equiv h(x).$$

If $q(x)$, $q_1(x)$, ... are the successive quotients and $r(x)$, $r_1(x)$, ... are the remainders in the process of repeated division applied to $f(x)$, $g(x)$,

$$f(x)\equiv g(x)q(x)+r(x)$$
$$g(x)\equiv r(x)q_1(x)+r_1(x)$$
$$r(x)\equiv r_1(x)q_2(x)+r_2(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$r_{k-2}(x)\equiv r_{k-1}(x)q_k(x)+r_k(x)$$

These identities show that each of the remainders is of the form $Af(x)+Bg(x)$ where A, B are polynomials in x. For example

$$r=A_0f+B_0g \text{ where } A_0=1,\ B_0=-q,$$
$$r_1=A_1f+B_1g \text{ where } A_1=-A_0q_1,\ B_1=1-B_0q_1,$$

and so on. But $h(x)$ is the last remainder $r_k(x)$ which does not vanish; hence taking $A\equiv A_k$, $B\equiv B_k$, $h(x)\equiv Af(x)+Bg(x)$.

This may be written $1\equiv AF(x)+BG(x)$ where $F(x)$, $G(x)$ are the polynomials obtained by dividing $f(x)$, $g(x)$ by their H.C.F. $h(x)$. It follows that $AF(x)+BG(x)$ has no algebraic factor, and therefore that A, B are co-prime.

In the special case when $f(x)$, $g(x)$ are co-prime, $h(x)\equiv r_k$ is a mere constant, and both sides of the identity may be divided by this constant. Hence

if $f(x)$, $g(x)$ are co-prime polynomials, other co-prime polynomials A, B exist such that $Af(x)+Bg(x)\equiv 1$.

The existence of A, B may be used to give a formal proof of the intuitive theorem:

If $F(x)$, $G(x)$ are co-prime polynomials and if G is a factor of $F(x)H(x)$, then $G(x)$ is a factor of $H(x)$.

For $AF+BG\equiv 1$, therefore $AFH+BGH\equiv H$.

But G is a factor of both AFH and BGH, hence it is a factor of H.

Also, *if F, G are co-prime and H is any other polynomial, then any common factor of FH and G is a factor of H.*

This also follows from the identity $AFH+BGH\equiv H$.

There is an important arithmetical theorem analogous to the algebraic theorem on p. 265:

If h is the H.C.F. of two positive integers f, g, then co-prime positive or negative integers A, B exist such that $Af+Bg=h$, and in particular if f, g are co-prime, integers A, B exist such that $Af+Bg=1$.

This may be proved by precisely the same argument as is used on p. 265. The following *alternative proof* of the algebraic theorem is instructive. See also p. 282, No. 23.

Consider the set of all possible polynomials $Af+Bg$ where A, B are polynomials such that $Af+Bg$ is not identically zero, and denote by R any one of them $A_1 f+B_1 g$ whose degree k is not greater than that of any other.

The special cases $A=1$, $B=0$ and $A=0$, $B=1$ show that k cannot exceed the degree of f or of g.

Let q, r be quotient and remainder when f is divided by R. Then the degree of r is less than k, and

$$r \equiv f-qR \equiv f-q(A_1 f+B_1 g) \equiv A_2 f+B_2 g.$$

Hence unless $r\equiv 0$ a new polynomial of the set has been found with degree less than k. This is inconsistent with the definition of k. Hence $r\equiv 0$. Thus R is a factor of f. Similarly R is a factor of g. But the identity $R\equiv A_1 f+B_1 g$ shows that any factor of f, g is a factor of R. Therefore R is the H.C.F. of f and g, that is $A_1 f+B_1 g\equiv h$.

Example 6. Find polynomials A, B such that

$$A(x^5+1)+B(x^2+x+1)\equiv 1.$$

In the method of repeated division, the work is

$$\begin{array}{r|l}
-x-1\ )\ x^2+x+1 & x^5+1 \qquad (\ x^3-x^2+1 \\
\underline{x^2+x} & \underline{x^5+x+1} \\
1 & -x
\end{array}$$

hence $$x^5+1=(x^2+x+1)(x^3-x^2+1)+(-x)$$

and $$x^2+x+1=(-x)(-x-1)+1.$$

Eliminating the term $(-x)$

$$(-x-1)(x^5+1)=(x^2+x+1)(x^3-x^2+1\ (-x-1)+(-x)(-x-1)$$
$$=(x^2+x+1)(-x^4+x^2-x-1)+(x^2+x+1)-1$$
$$\therefore\ (x+1)(x^5+1)+(-x^4+x^2-x)(x^2+x+1)=1,$$
$$\therefore\ A\equiv x+1,\ B\equiv -x^4+x^2-x.$$

In the identity $Af(x)+Bg(x)\equiv h(x)$, the polynomials A, B are not unique, for evidently

$$A'\equiv A+C\frac{g(x)}{h(x)} \qquad B'\equiv B-C\frac{f(x)}{h(x)}$$

where C is any polynomial, are also polynomials which satisfy the identity.

Conversely if $Af+Bg=h$ and $A'f+B'g=h$, A' and B' can be written in the forms $A+Cg/h$, $B-Cf/h$, where C is a polynomial.

For $$(A'-A)f=(B-B')g,$$
so $$(A'-A)f/h=(B-B')g/h.$$

But f/h, g/h are co-prime polynomials. Hence by the theorem on p. 265, f/h is a factor of $B-B'$.

Let $B-B'=Cf/h$, then $(A'-A)f=(B-B')g=Cfg/h$; so

$$A'-A=Cg/h.$$

Thus $$A'=A+Cg/h,\ B'=B-Cf/h.$$

Hence there exists *at most* one pair of polynomials A_1, B_1 of degrees less than those of g/h, f/h respectively, such that

$$A_1f+B_1g=h,$$

and it is easy to show that one such pair does exist.

For if $Af+Bg=h$ and if P, A_1 are the quotient and remainder when A is divided by g/h, and if Q, B_1 are the quotient and remainder when B is divided by f/h,

$$A=Pg/h+A_1,\ B=Qf/h+B_1.$$

Then $$h=Af+Bg=(P+Q)fg/h+A_1f+B_1g.$$

But the degrees of h, A_1f, B_1g are each less than that of fg/h, therefore $P+Q=0$ and $A_1f+B_1g=h$.

The functions A_1, B_1 may also be found by equating coefficients in an assumed identity, but this method is usually more laborious than the method used in Example 6.

EXERCISE XIIc

A

Find the simplest polynomials A, B which satisfy the identities in Nos. 1, 2, and give the general solution in polynomials.

1. $A(x^3+1)+B(x^2+2x+2)\equiv 1$.

2. $A(2x^4+x^3-16x^2-4x+2)+B(2x^3+5x^2-4x-3)\equiv 2x+1$

3. Find positive or negative integers A, B such that

$$17A+29B=1,$$

and give the general solution in integers.

4. If the polynomials ax^3+bx+c and px^3+qx+r are not co-prime, prove that $(br-cq)(aq-bp)^2=(cp-ar)^3$.

5. If a, b, c, d are constants such that $ad\neq bc$, and if f, g, r, s are polynomials in x such that $r\equiv af+bg$, $s\equiv cf+dg$, prove that the H.C.F. of f, g is the same as the H.C.F. of r, s. What is the conclusion if $ad=bc$?

B

Find the simplest polynomials A, B which satisfy the identities in Nos. 6-8.

6. $A(2x^3-4x^2+2x-3)+B(x^2-2x+3)\equiv 1$

7. $A(x^4+x^3-1)+B(x^2+1)\equiv x^2$

8. $A(3x^3-8x^2+19x-10)+B(3x^4+4x^3-22x^2-9x+14)\equiv 3x-2$

9. Find two integers A, B such that $106A+137B=1$.

10. Prove that the necessary and sufficient condition that two polynomials $f(x)$, $g(x)$ of degrees m, n are not co-prime is that polynomials $r(x)$, $s(x)$ of degrees less than m, n respectively exist such that $f(x)s(x)+g(x)r(x)=0$.

C

11. Prove that px^3+qx^2+rx+s, sx^3+rx^2+qx+p have a common factor of degree greater than unity if $p^2+qs=s^2+pr$.

12. If $a_1c_2+a_2c_1=2b_1b_2$ and $a_1x^2+2b_1x+c_1$, $a_2x^2+2b_2x+c_2$ have a common factor, prove that one of these polynomials is a perfect square. Interpret this result geometrically.

13. If $f(x)$, $g(x)$ are co-prime polynomials and if $r(x)$ is a polynomial of degree less than that of fg, prove that r can be expressed in the form $Af+Bg$, where A, B are polynomials of degrees less than those of g, f respectively.

Partial Fractions. A *rational function* of x is a function of the form $\frac{f(x)}{g(x)}$ where $f(x)$, $g(x)$ are polynomials in x. It is called *irreducible* if $f(x)$, $g(x)$ are co-prime, and *proper* if the degree of $f(x)$ is less than that of $g(x)$.

If $f(x)$, $g(x)$ have the H.C.F. $h(x)$, and

$$f(x)=h(x)f_1(x), \quad g(x)=h(x)g_1(x),$$

then $f/g=f_1/g_1$ for all values of x except those for which $g=0$.

If f/g is not proper, it can be expressed in the form $q+(r/g)$ as the sum of a polynomial and a proper rational function, by finding the quotient q and the remainder r when f is divided by g.

It will be assumed in the theorems that follow that the given rational functions are irreducible.

The process of expressing a given rational function in partial fractions is illustrated in Chapter V, pp. 89-93, by several numerical examples. The more general results which will be established in this chapter are required for the theory of integration.

If P_1, P_2 are co-prime polynomials, the rational function $\frac{A}{P_1P_2}$ can be expressed uniquely in the form $B+\frac{A_1}{P_1}+\frac{A_2}{P_2}$ where B is a polynomial and A_1/P_1, A_2/P_2 are proper and irreducible.

Since P_1, P_2 are co-prime, polynomials L_1, L_2 exist such that $L_1P_1+L_2P_2\equiv 1$,

hence
$$\frac{A}{P_1P_2}=\frac{A(L_1P_1+L_2P_2)}{P_1P_2}=\frac{AL_2}{P_1}+\frac{AL_1}{P_2}.$$

By division, $AL_2=P_1B_1+A_1$ $AL_1=P_2B_2+A_2$ where the degrees of A_1, A_2 are less than those of P_1, P_2 respectively.

Hence $\frac{A}{P_1P_2}=B_1+B_2+\frac{A_1}{P_1}+\frac{A_2}{P_2}$ where $\frac{A_1}{P_1}$, $\frac{A_2}{P_2}$ are proper.

A_1/P_1, A_2/P_2 must be irreducible. If for example A_1/P_1 reduces to A_1'/P_1', then the identity shows that $A/(P_1P_2)$ is equal to a fraction with denominator $P_1'P_2$, and this contradicts the assumption that $A/(P_1P_2)$ is irreducible.

To prove that the expression for $A/(P_1P_2)$ is unique, suppose that

$$B+\frac{A_1}{P_1}+\frac{A_2}{P_2}=\frac{A}{P_1P_2}=D+\frac{C_1}{P_1}+\frac{C_2}{P_2}.$$

Then $\qquad (B-D)P_1P_2+(A_1-C_1)P_2=(C_2-A_2)P_1$

$\therefore$ P_2 is a factor of $(C_2-A_2)P_1$ unless this is zero. But P_2 is of higher degree than C_2, A_2, and is therefore not a factor of C_2-A_2. Also it is prime to P_1. Hence $C_2\equiv A_2$, similarly $C_1\equiv A_1$ and $\therefore$ $B\equiv D$.

Thus the result is proved. Repeated applications of it show that

if $P_1, P_2, \ldots, P_n$ are polynomials every two of which are co-prime, the rational function $A/(P_1P_2\ldots P_n)$ can be expressed in the form

$$B+\frac{A_1}{P_1}+\frac{A_2}{P_2}+\ldots+\frac{A_n}{P_n}$$

where B is a polynomial and $A_1/P_1, \ldots, A_n/P_n$ are proper and irreducible.

The same argument as above will prove that this expression for $A/(P_1P_2\ldots P_n)$ is *unique*.

Application to the general Rational Function.

From p. 257, any polynomial of real algebra can be expressed as the product of factors like $(x-\alpha)^r$, $\{(x-\beta)^2+\gamma^2\}^s$, any two of the factors being co-prime. Hence by the theorem about

$$A/(P_1P_2\ldots P_n),$$

the general rational function $f(x)/g(x)$ can be expressed in the form

$$a+\Sigma\frac{B}{(x-\alpha)^r}+\Sigma\frac{C}{\{(x-\beta)^2+\gamma^2\}^s}$$

where a, B, C are polynomials and the degrees of B, C are less than r, $2s$ respectively.

By division, any polynomial $\phi(x)$ of degree m can be expressed in the form $(x-\alpha)\phi_1(x)+p_0$, where p_0 is a constant, and then $\phi_1(x)$ can be expressed in the form $(x-\alpha)\phi_2(x)+p_1$, and so on. This gives

$$\phi(x)=p_0+p_1(x-\alpha)+p_2(x-\alpha)^2+\ldots+p_m(x-\alpha)^m$$

where $p_0, p_1, \ldots, p_m$ are constants. Hence the proper fraction $\dfrac{B}{(x-\alpha)^r}$ may be replaced by

$$\frac{p_0}{(x-\alpha)^r}+\frac{p_1}{(x-\alpha)^{r-1}}+\ldots+\frac{p_{r-1}}{x-\alpha}.$$

Similarly by division any polynomial $\phi(x)$ can be expressed in the form $\{(x-\beta)^2+\gamma^2\}\phi_1(x)+q_0x+r_0$ where q_0, r_0 are constants, and then $\phi_1(x)$ can be expressed in the form $\{(x-\beta)^2+\gamma^2\}\phi_2(x)+q_1x+r_1$, and so on. If $\phi(x)$ is of degree $2s-1$ or $2s-2$, this gives

$$\phi(x)=(q_0x+r_0)+(q_1x+r_1)\{(x-\beta)^2+\gamma^2\}+\ldots$$
$$+(q_{s-1}x+r_{s-1})\{(x-\beta)^2+\gamma^2\}^{s-1}$$

where $q_0, r_0, q_1, r_1, \ldots, q_{s-1}, r_{s-1}$ are constants. Hence the proper fraction $\dfrac{C}{\{(x-\beta)^2+\gamma^2\}^s}$ may be replaced by

$$\frac{q_0x+r_0}{\{(x-\beta)^2+\gamma^2\}^s}+\frac{q_1x+r_1}{\{(x-\beta)^2+\gamma^2\}^{s-1}}+\ldots+\frac{q_{s-1}x+r_{s-1}}{(x-\beta)^2+\gamma^2}$$

By using the argument on p. 270, it may be proved that this reduction is unique. The final result

$$a+\sum\sum\frac{p_k}{(x-\alpha)^{r-k}}+\sum\sum\frac{q_kx+r_k}{\{(x-\beta)^2+\gamma^2\}^{s-k}}$$

shows what forms should be assumed in expressing a rational function as the sum of partial fractions. It remains to discuss methods more convenient than division of obtaining the values of the constants in numerical examples. See also Chapter V, pp. 89-93.

Example 7. Express $x^4+2x^3+3x^2+4x+5$ in the form

$$p_0+p_1(x-6)+p_2(x-6)^2+p_3(x-6)^3+p_4(x-6)^4.$$

Method 1. Divide repeatedly by $x-6$, using the method explained in Chapter VIII, p. 161.

1	2	3	4	5
	8	51	310	1865
	14	135	1120	
	20	255		
	26			

Hence the expression is

$$1865+1120(x-6)+255(x-6)^2+26(x-6)^3+(x-6)^4.$$

Method 2. Putting $x=\alpha$ in

$$\phi(x)\equiv p_0+p_1(x-\alpha)+p_2(x-\alpha)^2+\ldots+p_m(x-\alpha)^m$$

gives $\phi(\alpha)=p_0$, and putting $x=\alpha$ after differentiating r times gives $\phi^r(\alpha)=r!\,p_r$.

In this example, $\phi(x)=x^4+2x^3+3x^2+4x+5$,

$$\phi'(x)=4x^3+6x^2+6x+4,\quad \phi^2(x)=12x^2+12x+6,$$

$$\phi^3(x)=24x+12,\quad \phi^4(x)=24.$$

Hence
$$p_0=\phi(6)\ =6^4+2.6^3+3.6^2+4.6+5=1865,$$
$$p_1=\phi'(6)=4.6^3+6.6^2+6.6+4=1120,$$
$$2!\,p_2=\phi^2(6)=12.6^2+12.6+6=510,$$
$$3!\,p_3=\phi^3(6)=6.26,\ \text{and}\ \ 4!\,p_4=24.$$

Method 3. Equate coefficients of x^4, x^3, x^2, x^1, x^0.

$$1=p_4,\ \ 2=-4.6p_4+p_3,$$
$$3=6.6^2p_4-3.6p_3+p_2,\ \ 4=-4.6^3p_4+3.6^2p_3-2.6p_2+p_1,$$
$$5=6^4p_4-6^3p_3+6^2p_2-6p_1+p_0,$$

and solve these equations.

Example 8. Express $x^5+5x^4+4x^3+3x+1$ in the form

$$(q_0x+r_0)+(q_1x+r_1)(x^2+x+1)+(q_2x+r_2)(x^2+x+1)^2.$$

Divide twice in succession by x^2+x+1 using detached coefficients.

$$\begin{array}{r|l} & 1+4-1-3 \\ \hline 1+1+1 & 1+5+4+0+3+1 \\ & \quad\ \ 4+3+0 \\ & \qquad -1-4+3 \\ & \qquad\quad -3+4+1 \\ & \qquad\qquad\quad 7+4 \end{array} \qquad\qquad \begin{array}{r|l} & \quad\ \ 1+3 \\ \hline 1+1+1 & 1+4-1-3 \\ & \quad\ \ 3-2-3 \\ & \qquad -5-6 \end{array}$$

Hence the expression is

$$(7x+4)+(-5x-6)(x^2+x+1)+(x+3)(x^2+x+1)^2$$

Alternatively the constants may be found by equating coefficients.

The determination of the partial fractions of $f(x)/g(x)$ is particularly simple when there are no repeated factors in $g(x)$.

It is easier to deal with linear than with quadratic factors. By the use of complex algebra, quadratic factors can be avoided, but the corresponding partial fractions have to be combined in pairs and further reduction may be necessary.

Consider the fraction $\dfrac{f(x)}{(x-\alpha)h(x)}$, $h(\alpha) \neq 0$.

This is expressible as $\dfrac{p}{x-\alpha} + \dfrac{q(x)}{h(x)}$

where p is a constant and $p = \dfrac{f(x)}{h(x)} - (x-\alpha)\dfrac{q(x)}{h(x)}$.

Putting α for x, $p = f(\alpha)/h(\alpha)$.

This result justifies the following rule for finding the numerator p of a partial fraction $p/(x-\alpha)$ when $x-\alpha$ is not a repeated factor of the denominator :

Remove $x-\alpha$ from the denominator and substitute α for x in the remaining fraction.

Note. If $g(x) = (x-\alpha)h(x)$, then $h(\alpha) = g'(\alpha)$.

Applying the rule, the partial fractions of

$\dfrac{f(x)}{(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)}$ in which $\alpha_1, \alpha_2, \dots, \alpha_n$ are unequal,

are found to be

$$\frac{f(\alpha_1)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)}\frac{1}{x-\alpha_1} + \frac{f(\alpha_2)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\dots(\alpha_2-\alpha_n)}\frac{1}{x-\alpha_2} + \dots$$

Hence, multiplying by $(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$,

$$f(x) = \frac{(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\dots(\alpha_1-\alpha_n)}f(\alpha_1) + \frac{(x-\alpha_1)(x-\alpha_3)\dots(x-\alpha_n)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\dots(\alpha_2-\alpha_n)}f(\alpha_2) + \dots$$

Writing $A_1, A_2, \dots$ for $f(\alpha_1), f(\alpha_2), \dots$, this becomes *Lagrange's interpolation formula* for a polynomial of degree $n-1$ which takes the assigned values $A_1, A_2, \dots, A_n$ for $x = \alpha_1, \alpha_2, \dots, \alpha_n$.

Example 9. Express $\dfrac{x+7}{(x+1)(x-3)(x^2+x+3)}$ in partial fractions.

The expression $=\dfrac{p_1}{x+1}+\dfrac{p_2}{x-3}+\dfrac{qx+r}{x^2+x+3}$

and $\quad p_1=\dfrac{-1+7}{(-1-3)(1-1+3)}=-\dfrac{1}{2}, \; p_2=\dfrac{3+7}{(3+1)(9+3+3)}=\dfrac{1}{6}.$

Hence

$$x+7\equiv\{-\tfrac{1}{2}(x-3)+\tfrac{1}{6}(x+1)\}(x^2+x+3)+(qx+r)(x+1)(x-3);$$

from coefficients of x^3, $\quad 0=-\frac{1}{3}+q, \quad \therefore \; q=\frac{1}{3}$

and putting $x=0$, $\quad 7=\frac{9}{2}+\frac{1}{2}-3r, \quad \therefore \; r=-\frac{2}{3}$.

$$\therefore \text{ the expression} = -\frac{1}{2(x+1)}+\frac{1}{6(x-3)}+\frac{x-2}{3(x^2+x+3)}$$

Example 10. Express $\dfrac{x^4}{(x-a)(x^2+a^2)}$ in partial fractions.

The given expression is an improper fraction. By division or by inspection it can be written as

$$x+a+\frac{N}{(x-a)(x^2+a^2)}$$

where N is of degree less than 3. Hence

$$\frac{x^4}{(x-a)(x^2+a^2)}=x+a+\frac{p}{x-a}+\frac{qx+r}{x^2+a^2}.$$

By the rule, $p=\dfrac{a^4}{a^2+a^2}=\dfrac{1}{2}a^2$. Thus

$$x^4\equiv(x^2-a^2)(x^2+a^2)+\tfrac{1}{2}a^2(x^2+a^2)+(qx+r)(x-a)$$

$$\therefore \; -\tfrac{1}{2}a^2x^2+\tfrac{1}{2}a^4\equiv(qx+r)(x-a)$$

$$\therefore \; -\tfrac{1}{2}a^2(x+a)\equiv qx+r$$

and the expression $=x+a+\dfrac{a^2}{2(x-a)}-\dfrac{a^3(x+a)}{2(x^2+a^2)}$

EXERCISE XIId

A

1. Express $x^3 - 2x^2 + 3x - 1$ in ascending powers of $x - 2$.

2. Express $8x^4 - x^3 + x$ in ascending powers of $2x + 1$.

3. Express $5 - 4x + 3x^2 + 2x^4 - x^5$ in the form

$$a + bx + (c + dx)(1 - x + x^2) + (e + fx)(1 - x + x^2)^2$$

Find the partial fractions in Nos. 4-10.

4. $\dfrac{x^2 - 4}{(x-1)(x-3)}$ 5. $\dfrac{x^3}{(x-a)(x-b)}$ 6. $\dfrac{2x^2 + 3x + 4}{(x+1)(x^2+1)}$

7. $\dfrac{3x+1}{x^4 - 1}$ 8. $\dfrac{x}{(x-a)(x^2+bx+c)}$, $b^2 < 4c$

9. $\dfrac{2x^2 + 1}{(x^2+1)(x^2+x+1)}$ 10. $\dfrac{1 + 4x + 12x^2}{1 - 16x^4}$.

B

11. Express $x^3 + 3x^2 + 4x + 5$ in ascending powers of $x + 2$.

12. Express x^4 in ascending powers of $x - 1$.

13. Express $x^7 + 3x^6 + x + 1$ in the form

$$a + bx + z(c + dx) + z^2(e + fx) + z^3(g + hx)$$

where $z = x^2 - x + 1$.

Find the partial fractions in Nos. 14-22.

14. $\dfrac{x}{(x-a)(x-b)}$ 15. $\dfrac{x^2 - x + 1}{(x-1)(x^2+1)}$ 16. $\dfrac{x^4 + x^2}{x^3 - 1}$

17. $\dfrac{x^3}{(x-a)(x^2+b^2)}$ 18. $\dfrac{2x^3 - 3x - 4}{(x^2-1)(x^2+2x+2)}$

C

19. $\dfrac{x^3}{(x-a)(x-b)(x-c)}$ 20. $\dfrac{x^3}{(x-a)(x-b)}$

21. $\dfrac{x - x^3}{1 - 6x^2 + x^4}$ 22. $\dfrac{8}{x^6 - 1}$

Repeated Factors. Consider the fraction $\dfrac{f(x)}{(x-\alpha)^r g(x)}$ where $g(\alpha) \neq 0$. As on pp. 270, 271 this is expressible as

$$\frac{p_0}{(x-\alpha)^r} + \frac{p_1}{(x-\alpha)^{r-1}} + \ldots + \frac{p_{r-1}}{x-\alpha} + \frac{q(x)}{g(x)}$$

and so $\dfrac{f(x)}{g(x)} \equiv p_0 + p_1(x-\alpha) + \ldots + p_{r-1}(x-\alpha)^{r-1} + (x-\alpha)^r \dfrac{q(x)}{g(x)}$.

Putting α for x, $p_0 = \dfrac{f(\alpha)}{g(\alpha)}$ and so the value of p_0 can be written down by a similar rule to that given on p. 273 for unrepeated factors.

The simplified form of $\dfrac{f(x)}{(x-\alpha)^r g(x)} - \dfrac{p_0}{(x-\alpha)^r}$ when $p_0 = \dfrac{f(\alpha)}{g(\alpha)}$ is $\dfrac{\phi(x)}{(x-\alpha)^{r-1} g(x)}$ from which by the same rule $p_1 = \dfrac{\phi(\alpha)}{g(\alpha)}$.

The values of p_2, p_3, ... can be found in succession in the same way. But it is usually convenient in a numerical example when r is large to proceed otherwise.

Putting $x - \alpha = z$,

$$\frac{f(z+\alpha)}{g(z+\alpha)} = p_0 + p_1 z + \ldots + p_{r-1} z^{r-1} + z^r \frac{q(z+\alpha)}{g(z+\alpha)}$$

and the values of p_0, p_1, ... , p_{r-1} can be found by successive differentiation and substitution of zero for z. This amounts to finding the coefficients in the expansion of $f(z+\alpha)/g(z+\alpha)$ by Maclaurin's theorem. The values might also be found by division.

In dealing with a *repeated quadratic factor*, the following method, due to Horner, may be used.

If $x - \beta = y$, $\dfrac{C}{\{(x-\beta)^2+\gamma^2\}^s}$ can be expressed in the form $\dfrac{yg(y^2)+h(y^2)}{(y^2+\gamma^2)^s}$ where g, h are polynomials in y^2 of lower degree than s. If $y^2 = t$, the expression becomes

$$y\frac{g(t)}{(t+\gamma^2)^s} + \frac{h(t)}{(t+\gamma^2)^s}$$

and each term can be reduced by the methods already given for linear factors.

Example 11. Express $\dfrac{x^2-8x+9}{(x+1)(x-2)^3}$ in partial fractions, and find the coefficient of x^r in its expansion in ascending powers of x when $|x|<1$.

First Method

The expression $=\dfrac{a}{x+1}+\dfrac{f(x)}{(x-2)^3}$, and $a=\dfrac{1+8+9}{(-1-2)^3}=-\dfrac{2}{3}$.

Put $x-2=z$, then

$$\frac{x^2-8x+9}{x+1}=\frac{(z+2)^2-8(z+2)+9}{z+3}=\frac{-3-4z+z^2}{3+z}$$

$$=(-1-\tfrac{4}{3}z+\tfrac{1}{3}z^2)(1+\tfrac{1}{3}z)^{-1}$$

$$=(-1-\tfrac{4}{3}z+\tfrac{1}{3}z^2)\{1-\tfrac{1}{3}z+\tfrac{1}{9}z^3+z^3g(z)\}$$

$$=-1-z+\tfrac{2}{3}z^2+z^3h(z).$$

Hence the expression $=-\dfrac{2}{3(x+1)}-\dfrac{1}{(x-2)^3}-\dfrac{1}{(x-2)^2}+\dfrac{2}{3(x-2)}$

Second Method

$$x^2-8x+9\equiv A(x-2)^3+B(x+1)(x-2)^2+C(x+1)(x-2)+D(x+1).$$

Put $x=2$, $\therefore\ D=-1$,

$$\therefore\ x^2-7x+10\equiv A(x-2)^3+B(x+1)(x-2)^2+C(x+1)(x-2),$$

$$\therefore\ x-5\equiv A(x-2)^2+B(x+1)(x-2)+C(x+1).$$

Put $x=2$, $\therefore\ C=-1$,

$$\therefore\ 2x-4\equiv A(x-2)^2+B(x+1)(x-2),$$

$$\therefore\ 2\equiv A(x-2)+B(x+1).$$

Put $x=2$, $\therefore\ B=\frac{2}{3}$; $\therefore\ A=-\frac{2}{3}$; and this gives the same result as before. These two methods should be compared with the two methods given for the same example on p. 92.

In the expansion of

$$-\tfrac{2}{3}(1+x)^{-1}+\tfrac{1}{8}(1-\tfrac{1}{2}x)^{-3}-\tfrac{1}{4}(1-\tfrac{1}{2}x)^{-2}-\tfrac{1}{3}(1-\tfrac{1}{2}x)^{-1}$$

the coefficient of x^r, if $|x|<1$, is

$$-\tfrac{2}{3}(-1)^r+\tfrac{1}{8}\times\tfrac{1}{2}(r+1)(r+2)2^{-r}-\tfrac{1}{4}(r+1)2^{-r}-\tfrac{1}{3}2^{-r},$$

that is
$$-\tfrac{2}{3}(-1)^r+\tfrac{1}{3}(3r^2-3r-22)2^{-r-4}.$$

Example 12. Express $\dfrac{x-1}{(x-2)(x^2-4x+5)^3}$ in partial fractions.

$x^2-4x+5=(x-2)^2+1$. Put $x-2=y$, then

$$\text{the expression} = \frac{y+1}{y(y^2+1)^3}.$$

First Method.

$$\frac{y+1}{y(y^2+1)^3}=\frac{p}{y}+\frac{q(y)}{(y^2+1)^3} \text{ where } p=1$$

$$=\frac{1}{y}+\frac{-y^5-3y^3-3y+1}{(y^2+1)^3}$$

Use the method of Example 8, p. 272, to express $-y^5-3y^3-3y+1$ in the form $ay+b+(cy+d)(y^2+1)+(ey+f)(y^2+1)^2$

```
                     -1+0-2+0                              -1+0
1+0+1 ) -1+0-3+0-3+1                   1+0+1 ) -1+0-2+0
              0-2+0-3                                     0-1+0
                    0-1+1
```

$$\therefore\ -y^5-3y^3-3y+1\equiv(-y+1)+(-y)(y^2+1)+(-y)(y^2+1)^2,$$

and the expression $=\dfrac{1}{y}+\dfrac{1-y}{(y^2+1)^3}-\dfrac{y}{(y^2+1)^2}-\dfrac{y}{y^2+1}$.

Second Method (Horner's).

$$\frac{y+1}{y(y^2+1)^3}=\frac{1}{(y^2+1)^3}+y\frac{1}{y^2(y^2+1)^3}$$

$$=\frac{1}{t^3}+y\frac{1}{(t-1)t^3} \quad \text{where } y^2+1=t$$

$$=\frac{1}{t^3}-\frac{y}{t^3}\left(1+t+t^2+\frac{t^3}{1-t}\right)$$

$$=\frac{1-y}{t^3}-\frac{y}{t^2}-\frac{y}{t}-\frac{y}{1-t}$$

$$=\frac{1-y}{(y^2+1)^3}-\frac{y}{(y^2+1)^2}-\frac{y}{y^2+1}+\frac{1}{y}$$

Third Method.

Let $$\frac{y+1}{y(y^2+1)^3} \equiv \frac{A}{y} + \frac{By+C}{y^2+1} + \frac{Dy+E}{(y^2+1)^2} + \frac{Fy+G}{(y^2+1)^3}$$

then $$y+1 \equiv A(y^2+1)^3 + (By+C)y(y^2+1)^2 + (Dy+E)y(y^2+1) + (Fy+G)y.$$

Put $y=0$, $\therefore A=1$,

$$\therefore -y^6 - 3y^4 - 3y^2 + y \equiv (By+C)y(y^2+1)^2 + (Dy+E)y(y^2+1) + (Fy+G)y,$$

$$\therefore -y^5 - 3y^3 - 3y + 1 \equiv (By+C)(y^2+1)^2 + (Dy+E)(y^2+1) + Fy + G.$$

Put $y=i$, $\therefore -i+1 = Fi+G$; $\therefore F=-1$, $G=1$,

and $$-y^5 - 3y^3 - 2y \equiv (By+C)(y^2+1)^2 + (Dy+E)(y^2+1)$$

$$\therefore -y^3 - 2y \equiv (By+C)(y^2+1) + Dy + E.$$

Put $y=i$, $\therefore -i = Di+E$; $\therefore D=-1$, $E=0$,

and $$-y^3 - y \equiv (By+C)(y^2+1), \quad \therefore -y \equiv By + C.$$

This gives the same result as before, and since $x-2=y$,

$$\text{the expression} = \frac{3-x}{(x^2-4x+5)^3} + \frac{2-x}{(x^2-4x+5)^2} + \frac{2-x}{x^2-4x+5} + \frac{1}{x-2}.$$

EXERCISE XIIe

A

Find partial fractions for the expressions in Nos. 1-6.

1. $\dfrac{x^2-x+1}{(x-1)^3}$
2. $\dfrac{x+1}{(x-1)^2(x^2+1)}$.
3. $\dfrac{2x^2+3x+1}{(x-2)^3(x-3)}$
4. $\dfrac{x+3}{(x-1)(x^2+1)^2}$
5. $\dfrac{(x-1)(x+3)}{(x+1)(x^3+1)}$
6. $\dfrac{4}{(x+1)^2(x^2+1)^2}$

Find the coefficients of x^r in the expansions of the functions in Nos. 7-9, stating when the expansions are valid.

7. $\dfrac{3x^2-2}{4-x^2}$
8. $\dfrac{3x}{(1-2x)(1+x^2)}$
9. $\dfrac{x^2+2x+3}{(x^2+1)(x+1)^2}$

10. Express $\dfrac{x}{(x+2)(x^2+2x+2)^2}$ in partial fractions.

B

Find partial fractions for the expressions in Nos. 11-16.

11. $\dfrac{1}{x^3(x-1)}$ 12. $\dfrac{2x^2-4x+1}{(x-1)^2(x-2)}$ 13. $\dfrac{2x^2-5}{(x-1)(x+3)(x-2)^2}$

14. $\dfrac{1}{x^4(x^2+1)^2}$ 15. $\dfrac{x}{(x-1)^3(x-2)(x-3)}$ 16. $\dfrac{12x^4}{(x^2-1)(x^3-1)}$

Find the coefficients of x^r in the expansions of the functions in Nos. 17, 18, stating when the expansions are valid.

17. $\dfrac{1}{1+x+x^2}$ 18. $\dfrac{1-3x}{(1-2x)(1-x)^2}$

C

Find partial fractions for the expressions in Nos. 19-21.

19. $\dfrac{1}{(x-1)^3(x-2)^4}$ 20. $\dfrac{25}{(x^2+4)(x^2-1)^2}$

21. $\dfrac{2x^4-8x^3+3x^2-12x}{(x^2+1)^2(x^2+2)^2}$

22. Find the coefficient of x^r in the expansion of the function $\dfrac{1}{(1-x)(1-x^2)(1-x^3)}$ if $|x|<1$.

23. State what method you would use for expressing

$$\frac{x+3}{(x^2+4x+5)^3(x+1)^2}$$

in partial fractions.

24. Express $\dfrac{1}{(x-a)(x-b)(x-c)^2}$ in partial fractions if a, b, c are all unequal.

25. Express $\dfrac{x^{n+1}}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)}$ as the sum of a polynomial and partial fractions, if $\alpha_1, \alpha_2, \ldots$ are all unequal.

MISCELLANEOUS EXAMPLES

EXERCISE XIIf

A

1. Find the simplest polynomials P, Q in x such that
$$(x^4 - x^3 + 2x^2 - 3x - 3)P + (x^4 - x^2 - 2x - 1)Q = 0.$$

2. Prove that $x^n - px^r + q = 0$ has two equal roots if
$$\{p(n-r)/(qn)\}^n = \{(n-r)/(qr)\}^r$$

3. Prove that $(x-1)^4$ is a factor of
$$x^{2n} - n^2x^{n+1} + 2(n^2-1)x^n - n^2x^{n-1} + 1$$

Express in partial fractions Nos. 4-6.

4. $\dfrac{x^3}{(x+a)(x^2+b^2)}$ 5. $\dfrac{6x^5}{(x^2-1)(x^2-4)}$ 6. $\dfrac{9}{(1-2x)(1+x)^2}$

7. If n is an integer greater than 3, express the fractional part of $(1+x)^n/(2-x)^3$ in partial fractions

8. Prove that $\displaystyle\sum_{r=0}^{n} \frac{(-1)^{n+r}(n+r)!}{(r!)^2(n-r)!(x+r)} = \frac{(x-1)(x-2)\dots(x-n)}{x(x+1)(x+2)\dots(x+n)}$ and deduce that
$$\frac{(n+1)!}{1!\,2!\,(n-1)!} - \frac{(n+2)!}{2!\,3!\,(n-2)!} + \dots \text{ to } n \text{ terms} = 1.$$

9. If the polynomial $f(x)$ is divided by the product $g(x)$ of n unequal factors $(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$, prove that the remainder can be expressed in the form $\displaystyle\sum \frac{f(\alpha_r)}{g'(\alpha_r)} \cdot \frac{g(x)}{x-\alpha_r}$.

B

10. Find the condition that $ax^3 + bx^2 + c$, $ax^3 + bx + c$ should have a common factor.

11. Find the conditions that $ax^3 + 3bx^2 + 3cx + d = 0$ should have three equal roots.

12. Given that $-1 + \sqrt{3}$ is a root of $x^4 - 5x^3 - 13x^2 + 20x - 6 = 0$, find the other roots.

Express in partial fractions Nos. 13-17.

13. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}$ 14. $\dfrac{x^2+ax+b}{(x+c)^3}$

C

15. $\dfrac{4x^2}{(x-1)^3(x^2+1)^2}$ 16. $\dfrac{8}{x^3(x^5-x^4-x+1)}$

17. $\dfrac{1}{(x-a)^2(x^2-2bx+c)}$, $b^2<c$.

18. Prove that if m and n are positive integers

$$\frac{1}{n+1}-\binom{m}{1}\frac{1}{n+2}+\binom{m}{2}\frac{1}{n+3}-\ldots \text{ to } m+1 \text{ terms}=\frac{m!\,n!}{(m+n+1)!}$$

19. If $1/\{(x-a)^m(x-b)^n\}$, $a \neq b$, is expressed in partial fractions, prove that the coefficient of $1/(x-a)^{m-r}$ is

$$(-1)^r\frac{n(n+1)\ldots(n+r-1)}{(a-b)^{n+r}r!}.$$

20. If $g(x)\equiv(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ where no two factors are equal, and if $f(x)$ is a polynomial in x of degree $n-2$, prove that $\sum f(\alpha_r)/g'(\alpha_r)=0$.

21. If $g(x)\equiv(x-a)^2\phi(x)$ where $\phi(a)\neq 0$, and if

$$\frac{f(x)}{g(x)}=\frac{p_0}{(x-a)^2}+\frac{p_1}{x-a}+\frac{F(x)}{\phi(x)},$$

prove that

$$p_0=2f(a)/g^2(a) \text{ and } p_1=\tfrac{2}{3}\{3f'(a)g^2(a)-f(a)g^3(a)\}/\{g^2(a)\}^2$$

where $g^r(x)$ denotes the r^{th} derivative of $g(x)$.

22. If n is a positive integer prove that

$$\binom{n}{1}\frac{1}{x+n}+\binom{n}{2}\frac{1!}{(x+n)(x+n-1)}+\binom{n}{3}\frac{2!}{(x+n)(x+n-1)(x+n-2)}$$
$$+\ldots \text{ to } n \text{ terms}=\frac{1}{x+1}+\frac{1}{x+2}+\ldots+\frac{1}{x+n}$$

23. Prove the arithmetical theorem on p. 266 by the alternative method given for the corresponding algebraic result, degree being replaced by absolute value.

CHAPTER XIII

THEORY OF EQUATIONS

Positions of the Roots of an Equation in real Algebra.

If $f(x)$ is a polynomial, not zero for $x=x_1$ or x_2, then the number of roots of the equation $f(x)=0$ which lie between x_1 and x_2 is **odd** *if $f(x_1)$, $f(x_2)$ have* **opposite** *signs.*

For by Chapter XII, p. 257,

$$f(x) \equiv a_0(x-p_1)(x-p_2)\dots(x-p_k)g(x)$$

where $g(x)$ is a product of factors of the form $(x-q)^2+r^2$ or else is unity. Since $f(x_1)$, $f(x_2)$ have opposite signs and $g(x)$ is positive, the linear factors cannot all be absent and

$$(x_1-p_1)(x_1-p_2)\dots(x_1-p_k), \quad (x_2-p_1)(x_2-p_2)\dots(x_2-p_k)$$

must have opposite signs. But (x_1-p), (x_2-p) have the same sign unless p lies between x_1, x_2. Hence an odd number of $p_1, p_2, \dots, p_k$ must lie between x_1, x_2.

Conversely when this is given, $f(x_1)$, $f(x_2)$ must have opposite signs. Hence

the number of roots of $f(x)=0$ which lie between x_1 and x_2 is **even** *or zero if $f(x_1)$, $f(x_2)$ have the* **same** *signs.*

Note. Instead of the statement $f(x_1)$, $f(x_2)$ have the same or opposite signs, may be substituted $f(x_1)f(x_2) >$ or < 0.

If $f(x) \equiv x^n + a_1x^{n-1} + \dots + a_n = x^n\left(1+\frac{a_1}{x}+\dots+\frac{a_n}{x^n}\right)$, a sufficiently large number K can be chosen so that $f(x)$ has the same sign as x^n for $x>K$ and for $x<-K$. Thus $f(K)$, $f(-K)$ have opposite signs if n is odd and the same sign if n is even. Hence an equation of odd degree has an odd number of roots and therefore at least one root; and an equation of even degree has an even number of roots or no roots. These results were proved by another method on pp. 257, 258, and they may be illustrated graphically.

It should also be noted that if a_n is negative and n is even, the equation has at least one positive root and one negative root because the signs of $f(K), f(0), f(-K)$ are $+, -, +$.

Since $f(0)=a_n$ and $f(K)$ is positive, the number of positive roots is odd if and only if a_n is negative. $f(K), f(-K)$ are usually denoted by $f(\infty), f(-\infty)$.

Rolle's Theorem. It is fundamental in analysis that if a function $f(x)$ has a derivative at every point of the interval $a<x<b$ and is continuous up to the ends of the interval, and if $f(a)=0=f(b)$, then $f'(x)$ vanishes for at least one value of x in this interval.

The algebraic form of the theorem may be stated as follows.

If x_1, x_2 are consecutive roots of $f(x)=0$ where $f(x)$ is a polynomial, then $f'(x)=0$ has an odd number of roots between x_1 and x_2.

This applies even if x_1, x_2 are multiple roots.

Let $f(x)\equiv(x-x_1)^p(x-x_2)^q g(x)$ where, since x_1, x_2 are consecutive roots, $g(x)$ has the same sign for the interval from x_1 to x_2.

By logarithmic differentiation

$$\frac{f'(x)}{f(x)}=\frac{p}{x-x_1}+\frac{q}{x-x_2}+\frac{g'(x)}{g(x)}, \text{ where } p\geqslant 1,\ q\geqslant 1.$$

Hence $\quad f'(x)=(x-x_1)^{p-1}(x-x_2)^{q-1}h(x)$

where $\quad h(x)\equiv\{p(x-x_2)+q(x-x_1)\}g(x)+(x-x_1)(x-x_2)g'(x).$

Since $h(x_1)\equiv p(x_1-x_2)g(x_1)$ and $h(x_2)\equiv q(x_2-x_1)g(x_2)$ have opposite signs, it follows that $h(x)$, and therefore $f'(x)$, has an odd number of roots between x_1 and x_2.

It may be deduced from Rolle's Theorem with the help of the theorem on p. 283 that if $f(x)$ is of degree n, necessary and sufficient conditions for $f(x)=0$ to have n *unequal* roots are that $f'(x)$ has $n-1$ unequal roots say $\beta_1, \beta_2, \dots, \beta_{n-1}$ in ascending order, and that the signs of the series

$$f(-\infty), f(\beta_1), f(\beta_2), \dots, f(\beta_{n-1}), f(\infty)$$

which is called Rolle's series, are alternate. The reader should illustrate these results graphically.

The theorem on p. 283 shows that if there are n changes of sign in any series $f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_n)$ where $\gamma_1 < \gamma_2 < \ldots < \gamma_n$, $f(x) = 0$ has n unequal roots. But if in any special case it is easy to find the roots of $f'(x) = 0$, as in Example 1, it is quicker to use Rolle's series than to search for other suitable values of x.

Example 1. Find the range of values of k if the equation $f(x) \equiv x^4 - 14x^2 + 24x - k = 0$ has 4 unequal roots.

$$f'(x) \equiv 4x^3 - 28x + 24 \equiv 4(x-2)(x+3)(x-1).$$

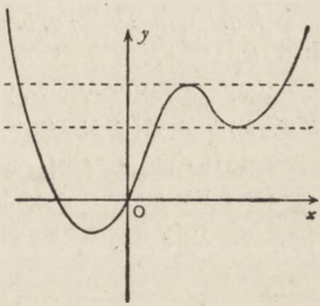
The roots of $f(x) = 0$ must be separated by those of $f'(x) = 0$, namely -3, 1, 2. The signs in Rolle's series

x	$-\infty$	-3	1	2	∞
$f(x)$	$+$	$-117-k$	$11-k$	$8-k$	$+$

must be +, −, +, −, +, and therefore

$$8 < k < 11.$$

The result may be illustrated by the graph of $y = x^4 - 14x^2 + 24x$ (not here drawn to scale). The straight line $y = k$ meets the curve in four distinct points if and only if $8 < k < 11$.



Example 2. Prove that the equation

$$f(x) \equiv \frac{a_1^2}{x-a_1} + \frac{a_2^2}{x-a_2} + \ldots + \frac{a_n^2}{x-a_n} + k = 0$$

has n roots if $a_1, a_2, \ldots, a_n$ are unequal.

First Method. Suppose that $a_1 < a_2 < \ldots < a_n$ and consider the polynomial $g(x) \equiv (x-a_1)(x-a_2)\ldots(x-a_n)f(x)$.

The series of signs in

x	$-\infty$	a_1	a_2	$\ldots$	a_{n-1}	a_n	∞
$g(x)$	$k(-1)^n$	$(-1)^{n-1}$	$(-1)^{n-2}$	$\ldots$	-1	$+$	k

contains n changes whether k is positive or negative and therefore $g(x) = 0$ has n roots.

Second Method. Consider the corresponding equation in complex algebra, $a_1, \ldots, a_n$ and k being x-axal, and denote any root by $p+qi$, then $p-qi$ is also a root. Therefore

$$\Sigma a_r^2\left(\frac{1}{p-qi-a_r}-\frac{1}{p+qi-a_r}\right)=0,$$

whence $\quad q\Sigma\dfrac{a_r^2}{(p-a_r)^2+q^2}=0$ and therefore $q=0$.

Thus all the roots are x-axal, and therefore in real algebra there are n roots.

Example 3. Prove that the equation

$$\phi(\lambda)\equiv(a-\lambda)(b-\lambda)(c-\lambda)+2fgh-(a-\lambda)f^2-(b-\lambda)g^2-(c-\lambda)h^2=0$$

has three roots.

If f, g, h are all zero, a, b, c are roots. Suppose then that $f\neq 0$ and write

$$\phi(\lambda)\equiv(a-\lambda)\{(b-\lambda)(c-\lambda)-f^2\}-\{(b-\lambda)g^2-2fgh+(c-\lambda)h^2\},$$
$$\psi(\lambda)\equiv(b-\lambda)(c-\lambda)-f^2.$$

Since the signs of $\psi(-\infty)$, $\psi(b)$, $\psi(\infty)$ are $+$, $-$, $+$, the equation $\psi(\lambda)=0$ has roots λ_1, λ_2 such that $\lambda_1<b<\lambda_2$.

But $\quad (b-\lambda)\{(b-\lambda)g^2-2fgh+(c-\lambda)h^2\}$
$$\equiv\{(b-\lambda)g-hf\}^2+h^2\{(b-\lambda)(c-\lambda)-f^2\},$$

thus
$$(b-\lambda_1)\phi(\lambda_1)=-\{(b-\lambda_1)g-hf\}^2$$
$$(b-\lambda_2)\phi(\lambda_2)=-\{(b-\lambda_2)g-hf\}^2.$$

If then $\phi(\lambda_1)\neq 0$, $\phi(\lambda_2)\neq 0$, the scheme of signs

λ	$-\infty$	λ_1	λ_2	∞
$\phi(\lambda)$	$+$	$-$	$+$	$-$

shows that $\phi(\lambda)=0$ has roots α, β, γ such that $\alpha<\lambda_1<\beta<\lambda_2<\gamma$.

If $\phi(\lambda_1)=0$ and $\phi(\lambda_2)\neq 0$, the signs are $+$, 0, $+$, $-$. Hence there is a root greater than λ_2 as well as the root λ_1. Similarly if $\phi(\lambda_1)\neq 0$ and $\phi(\lambda_2)=0$, there is a root less than λ_1 as well as the root λ_2. In these two cases and also when both λ_1 and λ_2 are roots, there must also be a third root because a cubic which has two roots must have a third.

Descartes' Rule of Signs. Some information about the number and position of the roots of an equation

$$f(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

can be derived from a consideration of the signs of the coefficients $a_1, a_2, \dots, a_n$. For convenience it is supposed that $a_n \neq 0$.

When two consecutive coefficients of a polynomial arranged in descending powers of x have the same sign there is said to be a *permanence* and when they have opposite signs there is said to be a *variation* in $f(x)$. For example $x^7 - 2x^5 - x^4 + 2x^2 - 7$ contains 1 permanence and 3 variations.

According as a_n is positive or negative, the number of variations in $f(x)$ is even or odd, but also from p. 284 the number of positive roots of $f(x) = 0$ is even or odd. Hence the number of variations of $f(x)$ and the number of positive roots of $f(x) = 0$ are of the same *parity*, that is they differ by an even number or zero.

The theorem of Descartes states that

if $f(x)$ is a polynomial with v variations, the number of positive roots of $f(x) = 0$ is not greater than v and is of the same parity as v.

The last part of this theorem has just been proved. The first part is proved by showing that if $g(x)$ is any polynomial and if k is positive, the expansion of $(x - k)g(x)$ contains at least one more variation than $g(x)$.

Arrange $g(x)$ in descending powers bracketing the permanences together. Then if $g(x)$ contains p variations,

$$g(x) \equiv (x^n + \dots + l_0 x^{n_1+1}) - (a_1 x^{n_1} + \dots + l_1 x^{n_2+1})$$
$$+ (-1)^2 (a_2 x^{n_2} + \dots + l_2 x^{n_3+1}) + \dots + (-1)^p (a_p x^{n_p} + \dots + c)$$

where $a_1, a_2, \dots, a_{p-1}, c$ are positive and the other constants are positive or zero.

In the expansion of $(x - k)g(x)$, the coefficient of x^{n_1+1} is $-kl_0 - a_1$ and is negative, and the coefficient of x^{n_r+1} has the sign of $(-1)^r$. Hence

$$(x - k)g(x) \equiv (x^{n+1} + \dots) - (b_1 x^{n_1+1} + \dots) + (-1)^2 (b_2 x^{n_2+1} + \dots)$$
$$+ \dots + (-1)^p (b_p x^{n_p+1} + \dots) + (-1)^{p+1} kc,$$

where the constants $k, c, b_1, b_2, \dots$ are positive and $(b_p x^{n_p+1} + \dots)$ must be replaced by cx if $a_p = 0$.

Thus whatever may be the signs of the terms not written down, the expansion of $(x-k)g(x)$ contains at least $p+1$ variations.

Suppose that $f(x)=0$ has exactly p positive roots. Then the equation is

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_p)(x^m+c_1x^{m-1}+\dots+c_m)=0$$

where $\alpha_1, \alpha_2, \dots, \alpha_p$ are positive. Therefore by what has just been proved $f(x)$ has at least p variations. Thus the number of positive roots of $f(x)=0$ cannot exceed the number of variations of $f(x)$. Since $\alpha_1, \alpha_2, \dots, \alpha_p$ are not necessarily unequal, an r-fold root must be counted as r roots in applying Descartes' theorem.

By writing $-x$ for x and applying the rule to the new equation, an upper limit is obtained for the number of negative roots of the original equation. Hence the total number of roots of $f(x)=0$ (excluding zero roots) cannot exceed the sum of the numbers of variations in $f(x)$ and $f(-x)$.

More generally if $y=(a-x)/(x-b)$ where $a>b$, then $y>0$ when $a>x>b$; hence if $f(x)\equiv h(y)$, the number of positive roots of $h(y)=0$ is the same as the number of roots of $f(x)=0$ between a and b. Thus Descartes' rule may be used to find an upper limit for the number of roots between a and b.

For a generalisation of Descartes' theorem, see Exercise XIIIb, No. 28.

Example 4. Find the number and positions of the roots of

$$f(x)\equiv 2x^5-4x^4-9x-2=0.$$

The number of variations in $f(x)$ is obviously the same as in $\frac{1}{2}f(x)\equiv x^5-2x^4-4\frac{1}{2}x-1$, so it is unnecessary to make the coefficient of the term of highest degree unity before using Descartes' rule.

$f(x)$ has 1 variation, $\therefore f(x)=0$ has at most 1 positive root.

$f(-x)\equiv -2x^5-4x^4+9x-2$, $\therefore f(-x)$ has 2 variations,

$$\therefore f(x)=0 \text{ has at most 2 negative roots.}$$

Also we have the table of values

x	-2	-1	0	2	3
$f(x)$	-112	1	-2	-20	133

$\therefore$ there are negative roots between -2, -1 and -1, 0, and one positive root between 2, 3.

Descartes' rule shows that there are no other roots.

Incomplete Equations. The equation

$$f(x) \equiv x^n + a_1 x^{n-1} + \dots + a_r x^{n-r} + \dots + a_n = 0$$

is called *complete* if all the constants $a_1, a_2, \dots, a_n$ are different from zero ; otherwise it is called *incomplete*.

Descartes' rule is more effective for incomplete equations. When applied to a complete equation, it can never prove that $f(x)=0$ has less than n roots because the permanences of $f(x)$ are variations of $f(-x)$, so that the sum of the number of variations of $f(x)$, $f(-x)$ is n.

Consider the incomplete equation

$$f(x) \equiv x^n + a_1 x^{n_1} + a_2 x^{n_2} + \dots + a_p x^{n_p} + a_{p+1} = 0$$

where $n > n_1 > n_2 > \dots > n_p > 0$, and the system of subsidiary equations

$$g_0(x) \equiv x^n + a_1 x^{n_1} = 0 \qquad g_1(x) \equiv a_1 x^{n_1} + a_2 x^{n_2} = 0$$
$$\dots\dots\dots\dots \qquad g_p(x) \equiv a_p x^{n_p} + a_{p+1} = 0.$$

If $n_r - n_{r+1}$ is even, $g_r(x) = 0$ has no roots or 2 roots other than zeros according as a_r, a_{r+1} have the same or opposite signs, and in these two cases the sum of the numbers of variations in $g_r(x)$, $g_r(-x)$ is also 0 or 2.

If $n_r - n_{r+1}$ is odd, $g_r(x) = 0$ has 1 non-zero root and the sum of the numbers of variations in $g_r(x)$, $g_r(-x)$ is also 1.

Hence the total number of non-zero roots of the subsidiary equations is equal to the sum of the numbers of variations of $f(x)$, $f(-x)$. In other words

an incomplete equation cannot have more roots than the total number of roots (other than zeros) of the subsidiary equations formed by equating each pair of consecutive terms to zero.

Example 5. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n$ and $a_r = a_{r+1} = a_{r+2}$, prove that $f(x) = 0$ cannot have more than $n - 2$ roots.

In the product $(x-1)f(x) \equiv x^{n+1} + b_1 x^n + \ldots + b_{n+1}$, $b_{r+1} = b_{r+2} = 0$, and therefore one of the subsidiary equations is

$$b_r x^{n-r+1} + b_{r+3} x^{n-r-2} = 0,$$

which has only one root other than zero.

Thus $(x-1)f(x) = 0$ cannot have more than $n - 1$ roots, and so $f(x) = 0$ cannot have more than $n - 2$ roots.

EXERCISE XIIIa

[*Throughout this exercise the algebra is real*]

A

Find the number of roots of the equations in Nos. 1, 2, and determine pairs of consecutive integers between which the roots lie.

1. $x^5 - 6x + 2 = 0$ 2. $x^6 - x^5 - 10x + 7 = 0$

3. If $x^4 + 4x^3 - 8x^2 + k = 0$ has 4 unequal roots, prove that $0 < k < 3$.

4. Find the range of values of k for which the equation $x^4 - 26x^2 + 48x - k = 0$ has 4 unequal roots.

5. If $p^2 > q$ and α, β are the roots of $x^2 - 2px + q = 0$, prove that $x^3 - 3px^2 + 3qx - r = 0$ has 3 roots if r lies between $pq - 2\alpha(p^2 - q)$ and $pq - 2\beta(p^2 - q)$.

6. If $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$, prove that

$$(x - a_1)(x - a_3)(x - a_5) + b^2(x - a_2)(x - a_4)(x - a_6) = 0$$

has 3 unequal roots, and generalise this result.

7. Prove that if m, n, p are unequal odd positive integers and if a, b, c are all unequal, the equation

$$f(x) \equiv a_1^2/(x-a)^m + b_1^2/(x-b)^n + c_1^2/(x-c)^p - 1 = 0$$

has exactly 3 roots.

8. If $fgh \neq 0$ and if two of the roots of the equation

$$(a-\lambda)(b-\lambda)(c-\lambda) - f^2(a-\lambda) - g^2(b-\lambda) - h^2(c-\lambda) + 2fgh = 0$$

are equal, prove that $a - gh/f = b - hf/g = c - fg/h$.

9. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n$, and if $a_r = 0$ and $a_{r+1} a_{r-1} > 0$, prove that $f(x) = 0$ has not more than $n - 2$ roots.

10. Explain the fallacy in the following argument:

If $\phi(x) \equiv \dfrac{A^2}{x-a} + \dfrac{B^2}{x-b} + \dfrac{C^2}{x-c} - k, \quad \phi'(x) = -\sum \dfrac{A^2}{(x-a)^2} < 0$;

$\therefore$ $\phi'(x) = 0$ has no roots; $\therefore$ by Rolle's theorem, $\phi(x) = 0$ has at most 1 root. But actually $\phi(x) = 0$ has 3 roots.

B

Find the number of roots of the equations in Nos. 11, 12, and determine pairs of consecutive integers between which the roots lie.

11. $x^3 - 7x + 2 = 0$ 12. $x^4 + x^3 - 10x - 3 = 0$

13. If $p \neq 0$, prove that $3x^4 + 4x^3 + p = 0$ has 2 roots or none according as $p < 1$ or $p > 1$.

14. Prove that $x(x-3)^2 = 4 \sin^2 \alpha$ has 3 positive roots, if $\alpha \neq n\pi$.

15. Prove that $x^3 + px + q = 0$ has 3 unequal roots if and only if $27q^2 + 4p^3 < 0$.

16. Prove that if $a_1, a_2, \ldots, a_n$ are all unequal, the equation $\sum 1/(x - a_r) = 0$ has $n - 1$ roots.

17. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n$ and if n is odd and $a_n > 0$, prove that $f(x) = 0$ has at least one negative root.

18. Prove that if $f(x)$ is a polynomial, the equation $f'(x) = 0$ has an even number of roots (or no root) greater than the greatest root of $f(x) = 0$.

19. If $f(x)$ is a polynomial and if $\beta_1, \beta_2, \beta_3$ are consecutive roots of $f'(x) = 0$, and if β_2 is an r-fold root where r is even, prove that if $f(\beta_2) \neq 0$, there is at most only one root of $f(x) = 0$ between β_1 and β_3.

C

Find the number of roots of the equations in Nos. 20, 21, and determine pairs of consecutive integers between which the roots lie.

20. $27x^4 - 45x^2 - 27x + 4 = 0$ 21. $x^6 - 3x^5 - 5x^2 + 2x + 9 = 0$

22. If $f(x) \equiv (x^2 - 1)^n$, prove that $f^n(x) = 0$ has n unequal roots.

23. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n$ and if $a_r^2 = a_{r-1} a_{r+1}$, prove that $f(x) = 0$ has not more than $n - 2$ roots.

24. If $p(p - n) > 0$, prove that the equation

$$x^n + px^{n-1} + \tfrac{1}{2}p(p-1)x^{n-2} + a_3 x^{n-3} + \ldots + a_n = 0$$

has not more than $n - 2$ roots.

Sturm's Theorem. This theorem determines exactly the number of roots of $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n = 0$ which lie between any two given numbers α and β.

If $\{\phi(x)\}^r$ is a factor of $f(x)$ and $r \geqslant 2$, $\phi(x)$ can be found by the method of p. 263, and the equation $\phi(x) = 0$ can be considered separately. *It is only necessary therefore to consider an equation $f(x) = 0$ which has no multiple roots.*

Let the process of finding by successive divisions the H.C.F. of $f(x)$ and its derivative $f'(x)$ be modified by changing the sign of each remainder before using it as a divisor, and let the remainders with their signs changed be $f_2(x), f_3(x), \ldots$.

For uniformity write $f_1(x)$ for $f'(x)$. Then the functions $f(x)$, $f_1(x), f_2(x), \ldots, f_m(x)$ are called Sturm's functions and we denote by $v(\xi)$ the number of variations of sign in the series of numbers $f(\xi), f_1(\xi), f_2(\xi), \ldots, f_m(\xi)$. We proceed to show that the number of roots of $f(x) = 0$ between α and β depends on $v(\alpha) - v(\beta)$.

The division process shows that each Sturm's function is of lower degree than the preceding one, so that $m \leqslant n$, and because it is assumed that there are no repeated roots, $f(x)$, $f_1(x)$ are co-prime and so the last function $f_m(x)$ is a constant and is not zero.

The only property of $f_m(x)$ that will be required is its invariable sign. If therefore an earlier function $f_k(x)$ is of invariable sign for $\alpha < x < \beta$, there is no need to calculate any function beyond $f_k(x)$.

Let the quotients in the H.C.F. process be $q_1, q_2, \ldots$. Then

$$f \equiv q_1 f_1 - f_2 \qquad f_1 \equiv q_2 f_2 - f_3$$
$$\cdots\cdots\cdots\cdots$$
$$f_{r-1} \equiv q_r f_r - f_{r+1}$$
$$\cdots\cdots\cdots\cdots$$
$$f_{k-2} \equiv q_{k-1} f_{k-1} - f_k$$

The Sturm's functions possess the following properties.

(i) Each of the functions $f, f_1, f_2, \ldots, f_k$ is a polynomial and *can only change sign* when x increases through a root, say $x = c$, of $f = 0$ or $f_r = 0$. If h is sufficiently small, f or f_r remains unaltered in sign for $c - h < x < c$ and for $c < x < c + h$.

(ii) *No two consecutive functions are both zero for the same value of x.*

For if when $x=c$, $f_{r-1}=0=f_r$, it follows from $f_{r-1}=q_r f_r - f_{r+1}$ that $f_{r+1}=0$ and from the next identity that $f_{r+2}=0$, and so on. Thus $f_k=0$, contrary to the definition of f_k.

(iii) f_{r-1} and f_{r+1} *have opposite signs for any value of x, say $x=c$, for which $f_r=0$*, and if h is sufficiently small, retain these signs throughout the interval $c-h<x<c+h$.

This is proved by the identity used in (ii) and by (i).

(iv) If $f(x)=0$ when $x=c$ and if h is positive and sufficiently small, *f and f_1 have opposite signs for $c-h<x<c$ and the same sign for $c<x<c+h$.*

Since $x=c$ is not a repeated root of $f(x)=0$, $f_1(c)\neq 0$ and h can be chosen so small that $f_1(c)$ has the same sign throughout the interval $c-h<x<c+h$. Also if h is sufficiently small, $f(x)$ has the same sign throughout $c-h\leqslant x<c$, and since

$$f(c-h)=f(c)-hf_1(c-\theta_1 h)=-hf_1(c-\theta_1 h) \quad (0<\theta_1<1,\ h>0)$$

this sign must be opposite to that of $f_1(c)$.

Similarly $f(c+h)=hf_1(c+\theta_2 h)$, $(0<\theta_2<1,\ h>0)$, and therefore $f(x)$ has the same sign as $f_1(c)$ in the interval $c<x<c+h$.

Consider now the change in the value of $v(x)$ when x increases from α to β.

By (i) there is no change in the value of $v(x)$ except possibly when x passes through a root of $f=0$ or of $f_r=0$.

By (iii) the series f_{r-1}, f_r, f_{r+1} contains exactly one variation on both sides of a root $x=c$ of $f_r=0$.

Thus the value of $v(x)$ can only change when x increases through a root $x=c$ of $f=0$; and by (iv), $v(x)$ is diminished by 1 when x increases through a root of $f=0$. Hence Sturm's theorem:

If $f(x)=0$ has no repeated roots and if α, β $(\alpha<\beta)$ are not roots of $f(x)=0$, the number of roots of $f(x)=0$ between α and β is $v(\alpha)-v(\beta)$, where $v(x)$ denotes the number of variations in the series $f, f_1, f_2, \cdots, f_k$ of Sturm's functions.

The reader should test his grasp of the argument used in this proof by showing that if β is a root of $f(x)=0$, the number of roots *between* α and β is $v(\alpha)-v(\beta)-1$ whether α is a root or not, and that if α is a root and β is not, the number is $v(\alpha)-v(\beta)$.

In numerical applications the work can often be simplified by multiplying Sturm's functions by *positive* constants to avoid fractions in the H.C.F. process, but multiplication by a negative constant would vitiate the result.

If the last Sturm function is a constant it often saves time to evaluate it or find its sign by using the remainder theorem.

Also if any function f_r is of the form $g_r(x)h_r(x)$ where $h_r(x)$ is positive throughout the interval $\alpha<x<\beta$, f_r may be replaced by g_r in continuing the H.C.F. process. The argument used to prove Sturm's theorem is not affected if after $f_{r-2}\equiv q_{r-1}f_{r-1}-f_r$ where $f_r\equiv g_rh_r$ and h_r is positive, we continue with $f_{r-1}\equiv q_rg_r-g_{r+1}$, taking g_{r+1} as the next Sturm's function, and so on.

Sturm's theorem gives a *necessary and sufficient* condition for an equation of degree n to have n *unequal* roots, since there must be a loss of n variations as x increases from $-\infty$ to $+\infty$. This is only possible if the series of Sturm's functions is complete, i.e. contains $n+1$ terms. Also the signs must be alternate for $x<-K$ and all the same for $x>K$, if K is sufficiently large.

Example 6. Find the number and positions of the roots of

$$f(x)\equiv x^4-4x^3+x^2+6x+2=0.$$

$$f'(x)\equiv 4x^3-12x^2+2x+6\equiv 2(2x^3-6x^2+x+3).$$

The H.C.F. process with detached coefficients is

2 − 6 + 1 + 3	2 − 8 + 2 + 12 + 4
	− 2 + 1 + 9 + 4
	− 5 + 10 + 7
10 − 30 + 5 + 15	5 − 10 − 7 $\equiv f_2$
− 10 + 19 + 15	− 5 − 7
− 1 + 1	− 12
$f_3\equiv$ 1 − 1	+ 12 $\equiv f_4$

∴ the series of Sturm's functions is

$$f \equiv x^4 - 4x^3 + x^2 + 6x + 2, \quad f_1 \equiv 2x^3 - 6x^2 + x + 3,$$
$$f_2 \equiv 5x^2 - 10x - 7, \quad f_3 \equiv x - 1, \; f_4 = 12.$$

For sufficiently large values of K the signs are

when $x < -K$,	+	−	+	−	+,	$v(-K)$	= 4,
when $x = 0$,	+	+	−	−	+,	$v(0)$	= 2,
when $x > K$,	+	+	+	+	+,	$v(K)$	= 0.

∴ there are 2 negative roots and 2 positive roots.

Further when $x = -1$, the signs are + − + − +, $v(-1) = 4$,
when $x = 2$, the signs are + − − + +, $v(2) = 2$,
when $x = 3$, the signs are + + + + +, $v(3) = 0$.

Therefore 2 roots lie between -1 and 0, and 2 roots lie between 2 and 3. It is left to the reader to separate each of these pairs of roots.

Note. When $x = 1$, we obtain + 0 − 0 +, giving 2 variations.

EXERCISE XIIIb

[Throughout this exercise the algebra is real]

Use Sturm's theorem for Nos. 1-17.

A

1. Prove that $x^3 - 7x + 7 = 0$ has 2 roots between 1 and 2, and 1 root between -3 and -4.

Find the number of positive roots and the number of negative roots of the equations in Nos. 2-5.

2. $x^4 - 22x^2 - 36x + 40 = 0$
3. $x^5 - 5x^3 + 25x + 1 = 0$
4. $x^4 - 3x^3 - 2x^2 + x - 3 = 0$
5. $x^n = x - 1$

6. Prove that $x^4 + 4rx + 3s = 0$ has no roots if $r^4 < s^3$.

7. Prove that if $p > 0$ and $q > 0$, $2x^5 - 5px^2 + 3q = 0$ has 1 root or 3 roots according as $q^3 >$ or $< p^5$.

B

8. Prove that $x^4 - 6x^3 + 14x^2 - 10x - 7 = 0$ has exactly 2 roots.

9. Find the number and positions of the roots of $x^5 - 5x + 1 = 0$.

Find the number of positive roots and the number of negative roots of the equations in Nos. 10-13.

10. $x^3 + x^2 - 18x + 10 = 0$

11. $x^4 + 4x^3 - 2x^2 + 8x + 2 = 0$

12. $3x^5 + 5x^3 - 2 = 0$

13. $x^4 - 2x^3 - 2k^2x + k^2 = 0$

C

14. Find the number of positive roots and the number of negative roots of $2x^n - nx^2 + 1 = 0$ where n is odd and greater than unity.

15. Prove that $x^4 + 2mx^3 - 2x/m - 1 = 0$ has exactly 2 roots.

16. Prove that $x^5 + 5ax^3 + b = 0$ has exactly 1 root if either $a > 0$ or $a < 0 < b^2 + 108a^5$.

17. Find the conditions that $x^n + ax + b = 0$ has 2 roots or no roots, given that n is even.

18. If the equation $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n = 0$ has n unequal roots and if $f, f_1, \ldots, f_r, \ldots$ is the series of Sturm's functions, prove that $f_r(x) = 0$ has $n - r$ unequal roots which separate the roots of $f_{r-1}(x) = 0$.

19. If the equation $f(x) = 0$ of degree n has n unequal roots, prove that $f(x)$ and its successive derivatives have the same properties as the Sturm's functions.

[*Nos. 20-28 are further applications of the methods of pp. 283-293*]

20. If the equation $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n = 0$ has n unequal roots, prove that (i) $f(x) + bf'(x) = 0$ has n unequal roots, (ii) $f(x)f^2(x) = \{f'(x)\}^2$ has no roots.

21. (i) If $f(x) \equiv x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \ldots = 0$ has n positive unequal roots, prove that $a_1, a_2, a_3, \ldots$ are all positive.

(ii) With the same notation, if $f(x) = 0$ has n unequal roots, prove that $a_1^2 - 2a_2 > 0$ and $a_2^2 - 2a_1 a_3 + 2a_4 > 0$.

22. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n = 0$ has n roots, prove that $a_r^2 > a_{r-1} a_{r+1}$.

23. Prove that if $f(x)$ is a polynomial, the equation

$$f(x) + af'(x) = 0$$

has at least as many roots as $f(x) = 0$.

Prove also that $f(x) + kaf'(x) + \binom{k}{2} a^2 f^2(x) + \ldots + \binom{k}{k} a^k f^k(x) = 0$ has at least as many roots as $f(x) = 0$.

24. If $f(x) \equiv (x - a_1)(x - a_2) \ldots (x - a_n)$ where $a_1 < a_2 < \ldots < a_n$, prove that $f'(x) = f(x) \sum 1/(x - a_r)$ and deduce that (i) the root of $f'(x) = 0$, which lies between a_1 and a_2, also lies between

$$a_1 + (a_2 - a_1)/n \text{ and } \tfrac{1}{2}(a_1 + a_2),$$

(ii) the root of $f'(x) = 0$, which lies between a_r and a_{r+1}, also lies between $a_r + (a_{r+1} - a_r)/n$ and $a_{r+1} - (a_{r+1} - a_r)/n$.

25. If $q(x)$ is the quotient and $r(x)$ the remainder when the polynomial $f(x)$ is divided by $f'(x)$ and if the roots of $f(x) = 0$ are unequal, prove that the roots of $q(x)r(x) = 0$ separate the roots of $f(x) = 0$.

Use this result to show that $x^4 - 8x^2 - 16x + 16 = 0$ has no negative roots and to find the positions of the positive roots.

26. If $f(x) \equiv (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$, prove that the Sturm's function $f_2(x)$ is $\frac{1}{n^2} \sum \frac{(\alpha_1 - \alpha_2)^2 f(x)}{(x - \alpha_1)(x - \alpha_2)}$, the α's being unequal.

27. If $f(x) \equiv (x - \alpha)^r g(x)$ where $g(x)$ is a polynomial and $g(\alpha) \neq 0$, the signs of $f(x)$ and its successive derivatives

$$f^1(x), f^2(x), \ldots, f^{r-1}(x)$$

alternate for $x = \alpha - \epsilon$, and are all the same as the sign of $f^r(\alpha)$ for $x = \alpha + \epsilon$, if ϵ is sufficiently small and positive.

28. [*Fourier's Theorem*] If $f(x)$ is a polynomial of degree n and if $f^r(x)$ denotes its r^{th} derivative, and if $v(\xi)$ denotes the number of variations in the signs of the series $f(x), f^1(x), f^2(x), \ldots, f^n(x)$, prove that the number of roots of $f(x) = 0$ between $x = \alpha$, $x = \beta$, $(\alpha < \beta)$, is not greater than $v(\alpha) - v(\beta)$ and has the same parity.

Symmetric Functions of the Roots in Complex Algebra.

In complex algebra, the equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \quad (a_0 \neq 0)$$

has n roots, and it was proved on p. 154 that the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy the relations

$$\textstyle\sum_1 = -a_1/a_0 \quad \sum_2 = a_2/a_0 \quad \ldots \quad \sum_r = (-1)^r a_r/a_0 \ldots$$

where $\sum_r$ is the sum of the products of the roots taken r at a time.

Example 7. If $\alpha, \beta, \gamma, \ldots$ are the n roots of

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$$

find the values of (i) $\sum\alpha^2$, (ii) $\sum\alpha^2\beta$, (iii) $\sum\alpha^{-2}$, (iv) $\sum\alpha\beta^{-2}$

(i) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta = a_1^2 - 2a_2$.

(ii) In the product $(\sum\alpha)(\sum\alpha\beta)$ the term $\alpha\beta\gamma$ occurs three times: as $\alpha.\beta\gamma$, $\beta.\gamma\alpha$, $\gamma.\alpha\beta$, and so

$$(\textstyle\sum\alpha)(\sum\alpha\beta) = \sum\alpha^2\beta + 3\sum\alpha\beta\gamma,$$

$$\therefore \textstyle\sum\alpha^2\beta = (-a_1)(a_2) - 3(-a_3) = 3a_3 - a_1 a_2.$$

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}, \ldots$ are the roots (see p. 160) of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + 1 = 0 \quad (a_n \neq 0)$$

i.e. of $$x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \ldots + \frac{a_1}{a_n} x + \frac{1}{a_n} = 0;$$

$$\therefore \text{ from (i) } \sum\frac{1}{\alpha^2} = \left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right) = (a_{n-1}^2 - 2a_n a_{n-2})/a_n^2.$$

(iv) $(\sum\alpha)(\sum\beta^{-2}) = \sum\alpha\beta^{-2} + \sum\alpha^{-1}$

$$\therefore \text{ from (iii) } \textstyle\sum\alpha\beta^{-2} = -a_1(a_{n-1}^2 - 2a_n a_{n-2})/a_n^2 + a_{n-1}/a_n$$
$$= (2a_1 a_{n-2} a_n - a_1 a_{n-1}^2 + a_{n-1} a_n)/a_n^2.$$

Example 8. If α, β, γ are the roots of $x^3 + a_1 x^2 + a_2 x + a_3 = 0$, find the value of $\sum(\alpha^3\beta^2)$.

In the product $(\sum\alpha^2\beta)(\sum\alpha\beta)$, $\alpha^3\beta\gamma$ occurs as $\alpha^2\beta.\alpha\gamma$ and as $\alpha^2\gamma.\alpha\beta$; also $\alpha\beta^2\gamma^2$ occurs as $\beta^2\gamma.\alpha\gamma$ and as $\beta\gamma^2.\alpha\beta$;

$$\therefore (\textstyle\sum\alpha^2\beta)(\sum\alpha\beta) = \sum\alpha^3\beta^2 + 2\sum\alpha^3\beta\gamma + 2\sum\alpha\beta^2\gamma^2$$

$$\therefore \textstyle\sum\alpha^3\beta^2 = (\sum\alpha^2\beta)(\sum\alpha\beta) - 2\alpha\beta\gamma(\sum\alpha^2 + \sum\beta\gamma)$$

∴ using the results of Example 7 (i) and (ii)

$$\sum\alpha^3\beta^2 = (3a_3 - a_1a_2)a_2 - 2(-a_3)(a_1^2 - a_2)$$
$$= a_2a_3 - a_1a_2^2 + 2a_1^2a_3$$

Weight and Order. An advantage of the suffix notation for the coefficients in the general equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

is that the suffix shows the degree in *all* the roots of the coefficient. For example $a_3 = -\sum\alpha\beta\gamma$ and this is a symmetric function of degree 3 in all the roots.

Hence if a rational integral symmetric function of the roots is expressed in terms of the coefficients, the sum of the suffixes of each term is equal to the degree of the symmetric function. This sum is called the **weight** of the function.

In Example 8, $\sum\alpha^3\beta^2 = a_2a_3 - a_1a_2^2 + 2a_1^2a_3$. The weight of this function is 5. It is the degree of each term of $\sum\alpha^3\beta^2$ and is the sum of the suffixes of each term on the right.

A consideration of the weight provides a quick check on the accuracy of a result. Another such check arises as follows: each coefficient when expressed in terms of the roots contains each root to the first degree or not at all; therefore no term in the function of coefficients can be of greater degree than the highest power to which any root occurs in the corresponding symmetric function of roots. This highest power is called the **order** of the function.

In Example 8, the order of $\sum\alpha^3\beta^2$ is 3, and the degrees of a_2a_3, $a_1a_2^2$, $a_1^2a_3$ are 2, 3, 3. Since $\sum\alpha^3\beta^2$ is of order 3 there cannot be a term like $a_1^3a_2$ because this is of degree 4, although it is of the same weight as $\sum\alpha^3\beta^2$.

These principles are not merely useful as checks but may be used to obtain the expression for a symmetric function in terms of the coefficients.

Example 9. If α, β, γ, δ are the roots of

$$x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

express the function

$$(\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2$$

in terms of a_1, a_2, a_3, a_4.

The function is of weight 4 and order 2 and is therefore of the form $\lambda a_1a_3 + \mu a_2{}^2 + \nu a_4$ where λ, μ, ν are numerical constants,

From the equation $(x+1)^4 = 0$ for which

$$a_1 = a_2 = a_3 = a_4 = 1 \text{ and } \alpha = \beta = \gamma = \delta = -1, \quad \lambda + \mu + \nu = 0.$$

From the equation $x^4 - x^2 = 0$ for which

$$a_1 = a_3 = a_4 = 0, \quad a_2 = -\tfrac{1}{6} \text{ and } \alpha = \beta = 0, \quad \gamma = 1, \quad \delta = -1, \quad \mu = 72.$$

Similarly from the equation $(x^2-1)^2 = 0$, $\tfrac{1}{9}\mu + \nu = 32$.

Hence $\nu = 24$, $\lambda = -96$, and the function is

$$24(a_4 - 4a_1a_3 + 3a_2{}^2).$$

Note. It is assumed that the function can be expressed in terms of the coefficients. This is proved for the general symmetric function on p. 304.

Example 10. If α, β, γ are the roots of $x^3 + 3Hx + G = 0$, form the equation whose roots are $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$, and evaluate $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$.

Put $y = (\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma = \alpha^2 + 4G/\alpha$ since $\alpha+\beta+\gamma = 0$ and $\alpha\beta\gamma = -G$.

Thus $\alpha y = \alpha^3 + 4G$. But $\alpha^3 + 3H\alpha + G = 0$

$$\therefore \alpha y = -3H\alpha + 3G, \quad \alpha = 3G/(y+3H)$$

$$\therefore 27G^3/(y+3H)^3 + 9GH/(y+3H) + G = 0.$$

Thus $(\beta-\gamma)^2$ and similarly $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$ are the roots of

$$27G^3 + 9GH(y+3H)^2 + G(y+3H)^3 = 0$$

which reduces to

$$y^3 + 18Hy^2 + 81H^2y + 27(G^2 + 4H^3) = 0.$$

Hence $\quad (\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27(G^2 + 4H^3).$

Example 11. If α, β, γ are the roots of $x^3+px+q=0$, form the equation whose roots are α^5, β^5, γ^5.

Put $y=\alpha^5$. Then $y^{3/5}+py^{1/5}=-q$.

To rationalise this equation we use the identity

$$(a+b)^5=a^5+b^5+5(a^4b+ab^4)+10(a^3b^2+a^2b^3)$$
$$=a^5+b^5+5ab(a+b)(a^2-ab+b^2)+10a^2b^2(a+b)$$
$$=a^5+b^5+5ab(a+b)\{a^2+b(a+b)\}.$$

Thus
$$-q^5=y^3+p^5y+5py^{4/5}(-q)(y^{6/5}-pqy^{1/5})$$
$$=y^3+p^5y-5pqy^2+5p^2q^2y.$$

Therefore α^5, β^5, γ^5 are the roots of the cubic

$$y^3-5pqy^2+p^2(p^3+5q^2)y+q^5=0.$$

EXERCISE XIIIc

[*Throughout this exercise the algebra is complex*]

A

1. If α, β, γ are the roots of $x^3+px+q=0$, form the equations whose roots are

(i) $\beta+\gamma-2\alpha,\ \gamma+\alpha-2\beta,\ \alpha+\beta-2\gamma$,

(ii) $1/\beta+1/\gamma,\ 1/\gamma+1/\alpha,\ 1/\alpha+1/\beta$.

2. If α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$, find the values of (i) $\sum\alpha^2\beta^2$, (ii) $\sum\alpha(\beta-\gamma)^2$

3. If α, β, γ, δ are the roots of $x^4+a_1x^3+a_2x^2+a_3x+a_4=0$, find the values of (i) $\sum\alpha^2\beta\gamma$, (ii) $\sum\alpha^3\beta$.

4. If α, β, γ, ... are the n roots of $x^n+a_1x^{n-1}+\ldots+a_n=0$, find the values of (i) $\sum(\alpha-\beta)^2$, (ii) $\sum\alpha/\beta$, (iii) $\sum\alpha^2/\beta$.

5. If α, β, γ, δ are the roots of

$$x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

find the value of $\sum\{(\alpha-\beta)^2(\gamma^2+\gamma\delta+\delta^2)\}$

B

6. If α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$, find the values of (i) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$, (ii) $\sum(\beta-\gamma)^2$, (iii) $\sum\alpha/\beta$.

7. If α, β, γ, ... are the n roots of $x^n+a_1x^{n-1}+\ldots+a_n=0$, find the values of (i) $\sum\alpha^2\beta\gamma$, (ii) $\sum\alpha^2\beta^2$.

8. If α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$, form the equation whose roots are $\beta+\gamma, \gamma+\alpha, \alpha+\beta$.

9. If α, β, γ are the roots of $x^3+px+q=0$, form the equation whose roots are $\beta^3+\gamma^3, \gamma^3+\alpha^3, \alpha^3+\beta^3$.

10. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$, find the value of $(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)$.

C

11. If α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$, find the value of $(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$.

12. If α, β, γ are the roots of $x^3+px+q=0$, form the equations whose roots are

(i) $\beta/\gamma+\gamma/\beta, \gamma/\alpha+\alpha/\gamma, \alpha/\beta+\beta/\alpha$; (ii) $\beta/\gamma, \gamma/\beta, \gamma/\alpha, \alpha/\gamma, \alpha/\beta, \beta/\alpha$.

13. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+a_1x^3+a_2x^2+a_3x+a_4=0$, find the value of $(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta)(\alpha\beta+\gamma\delta)$ and form the equation whose roots are $\beta\gamma+\alpha\delta, \gamma\alpha+\beta\delta, \alpha\beta+\gamma\delta$.

14. If $\alpha, \beta, \gamma, \dots$ are the n roots of $x^n+a_1x^{n-1}+\dots+a_n=0$, find the value of $\sum\alpha^2\beta^2\gamma$.

15. If α, β, γ are the roots of $x^3+a_1x^2+a_2x+a_3=0$, form the equations whose roots are

$$\text{(i) } \alpha^2-\beta\gamma,\ \beta^2-\gamma\alpha,\ \gamma^2-\alpha\beta,$$
$$\text{(ii) } \beta^2+\gamma^2-\alpha^2,\ \gamma^2+\alpha^2-\beta^2,\ \alpha^2+\beta^2-\gamma^2.$$

16. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$, find the values of

(i) $\sum\alpha\beta(\gamma+\delta)^3$, (ii) $(\alpha+\beta)(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)(\gamma+\delta)$.

Newton's Formula for Sums of Powers of Roots.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=0.$$

Denote $\alpha_1^r+\alpha_2^r+\dots+\alpha_n^r$ by s_r, and for convenience suppose that $0=a_{n+1}=a_{n+2}=\dots$

Then by logarithmic differentiation of

$$\phi(y)\equiv 1+a_1y+a_2y^2+\dots+a_ny^n\equiv(1-\alpha_1y)(1-\alpha_2y)\dots(1-\alpha_ny)$$

$$\frac{\phi'(y)}{\phi(y)}=-\sum\frac{\alpha}{1-\alpha y}=-\sum\left(\alpha+\alpha^2y+\dots+\alpha^py^{p-1}+\frac{\alpha^{p+1}y^p}{1-\alpha y}\right)$$
$$=-(s_1+s_2y+\dots+s_py^{p-1})-y^p\sum\{\alpha^{p+1}/(1-\alpha y)\}$$

Thus $\phi'(y) + (s_1 + s_2 y + \ldots + s_p y^{p-1})\phi(y)$

$$\equiv b_p y^p + b_{p+1} y^{p+1} + \ldots + b_{p+n-1} y^{p+n-1}$$

Therefore the coefficient of y^r in

$$a_1 + 2a_2 y + \ldots + na_n y^{n-1} + (s_1 + s_2 y + \ldots + s_p y^{p-1})(1 + a_1 y + a_2 y^2 + \ldots + a_n y^n)$$

is zero for $r = 0, 1, 2, \ldots, p-1$. Thus

$$s_1 + a_1 = 0$$
$$s_2 + s_1 a_1 + 2a_2 = 0$$
$$s_3 + s_2 a_1 + s_1 a_2 + 3a_3 = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_r + s_{r-1} a_1 + s_{r-2} a_2 + \ldots + s_1 a_{r-1} + r a_r = 0.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and as p may be an integer as large as we please, this result is true for *all* values of r taking into account the convention

$$0 = a_{n+1} = a_{n+2} = \ldots$$

It is called *Newton's formula* for the powers of the roots.

Newton's equations give in succession the values of $s_1, s_2, s_3, \ldots$, in terms of the coefficients, and show that s_r *is an integral function of* $a_1, a_2, \ldots, a_r$.

To find s_r directly, a determinant may be used to eliminate $s_1, s_2, \ldots, s_{r-1}$. See p. 180. For example from the first three equations

$$\begin{vmatrix} s_3 + 3a_3 & a_2 & a_1 \\ 2a_2 & a_1 & 1 \\ a_1 & 1 & 0 \end{vmatrix} = 0, \text{ thus } s_3 = \begin{vmatrix} 3a_3 & a_2 & a_1 \\ 2a_2 & a_1 & 1 \\ a_1 & 1 & 0 \end{vmatrix}$$

Also the value of $\alpha_1^{-r} + \alpha_2^{-r} + \ldots + \alpha_n^{-r}$ can be found from the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + 1 = 0$ whose roots are $\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1}$.

Example 12. If $\alpha + \beta + \gamma + \delta = 0$, prove that

$$\tfrac{1}{5}(\alpha^5 + \beta^5 + \gamma^5 + \delta^5) = \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \times \tfrac{1}{3}(\alpha^3 + \beta^3 + \gamma^3 + \delta^3).$$

The equation whose roots are $\alpha, \beta, \gamma, \delta$ is of the form

$$x^4 + a_2 x^2 + a_3 x + a_4 = 0.$$

From Newton's equations since $s_1 = 0$ and $a_1 = 0$,

$$s_2 + 2a_2 = 0 \quad s_3 + 3a_3 = 0 \quad s_5 + s_3 a_2 + s_2 a_3 = 0$$

$$\therefore\ s_5 = -s_3(-\tfrac{1}{2}s_2) - s_2(-\tfrac{1}{3}s_3) = \tfrac{5}{6}s_2 s_3.$$

From Newton's equations may be deduced the important theorem :

Every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients.

This was assumed in Example 9, p. 300.

It is sufficient to consider $\sum(\alpha^p\beta^q\gamma^r\ldots)$, since any rational symmetric function can be expressed as a fraction whose numerator and denominator are sums of terms of this type.

For the *double* function $\sum\alpha^p\beta^q \quad (p \neq q)$

$$(\sum\alpha^p)(\sum\beta^q) = \sum\alpha^{p+q} + \sum\alpha^p\beta^q$$

$$\therefore \ \sum\alpha^p\beta^q = s_p s_q - s_{p+q}$$

Next for the *triple* function $\sum\alpha^p\beta^q\gamma^r$, p, q, r being all unequal,

$$(\sum\alpha^p\beta^q)(\sum\gamma^r) = \sum\alpha^{p+r}\beta^q + \sum\alpha^p\beta^{q+r} + \sum\alpha^p\beta^q\gamma^r,$$

and so the triple function can be expressed in terms of s_r and double functions.

Similarly a quadruple function can be derived from triple functions, and so on. The general result then follows by induction.

Certain modifications are necessary in these formulae if the indices are not all unequal. For example

$$(\sum\alpha^p)^2 = \sum\alpha^{2p} + 2\sum\alpha^p\beta^p \text{ ; thus } \sum\alpha^p\beta^p = \tfrac{1}{2}(s_p{}^2 - s_{2p}).$$

Similarly the formula

$$\sum\alpha^p\beta^q\gamma^r = s_p s_q s_r - s_{q+r}s_p - s_{r+p}s_q - s_{p+q}s_r + 2s_{p+q+r}$$

which may be deduced from the identity given above where p, q, r are all unequal, is replaced by

$$\sum\alpha^p\beta^p\gamma^r = (s_p{}^2 s_r - 2s_{p+r}s_p - s_{2p}s_r + 2s_{2p+r})/2!$$

and $$\sum\alpha^p\beta^p\gamma^p = (s_p{}^3 - 3s_{2p}s_p + 2s_{3p})/3!$$

and so on.

Newton's equations also give in succession the values of $a_1, a_2, a_3, \ldots$ in terms of $s_1, s_2, s_3, \ldots$ and show that *a_r is a rational integral function of $s_1, s_2, \ldots, s_r$.*

Hence, regarding the coefficients as functions of $s_1, s_2, s_3, \ldots$

$$\frac{\partial a_p}{\partial s_r}=0 \quad \text{if } p<r$$

because a_p does not involve s_r if $p<r$. And from the equation

$$s_r+s_{r-1}a_1+\ldots+s_1a_{r-1}+ra_r=0$$

by partial differentiation with respect to s_r

$$1+r\frac{\partial a_r}{\partial s_r}=0, \quad \therefore \quad \frac{\partial a_r}{\partial s_r}=-\frac{1}{r}.$$

Similarly from the next equation,

$$a_1+s_1\frac{\partial a_r}{\partial s_r}+(r+1)\frac{\partial a_{r+1}}{\partial s_r}=0;$$

but $s_1=-a_1$ and $\dfrac{\partial a_r}{\partial s_r}=-\dfrac{1}{r}$, $\quad \therefore \quad \dfrac{\partial a_{r+1}}{\partial s_r}=-\dfrac{a_1}{r}$.

And it is easy to prove by induction (see Exercise XIIId, No. 12) that

$$\frac{\partial a_{r+k}}{\partial s_r}=-\frac{a_k}{r} \quad (1\leqslant k\leqslant n-r).$$

Example 13. If $\alpha, \beta, \gamma, \ldots$ are the n roots of

$$x^n+a_1x^{n-1}+\ldots+a_n=0 \quad (n\geqslant 5)$$

find the value of $\sum\alpha^3\beta^2$.

$\sum\alpha^3\beta^2$ is of order 3 and weight 5. Hence by using the result of Example 8, p. 298, for $x^{n-3}(x^3+a_1x^2+a_2x+a_3)=0$,

$$\sum\alpha^3\beta^2=a_2a_3-a_1a_2^2+2a_1^2a_3+\lambda a_1a_4+\mu a_5.$$

From p. 304, $\sum\alpha^3\beta^2=s_3s_2-s_5$.

Differentiate in turn partially with respect to s_5, s_4,

$$\therefore \quad -1=\mu(-\tfrac{1}{5}), \quad \therefore \quad \mu=5$$

and $\quad 0=\lambda a_1(-\tfrac{1}{4})+\mu a_1(-\tfrac{1}{4})=0, \quad \therefore \quad \lambda=-\mu=-5.$

$$\therefore \quad \sum\alpha^3\beta^2=a_2a_3-a_1a_2^2+2a_1^2a_3-5a_1a_4+5a_5.$$

EXERCISE XIIId

[*Throughout this exercise the algebra is complex and s_r denotes the sum of the rth powers of the roots of the given equation*]

A

1. Form the cubic for which $s_1=3$, $s_2=5$, $s_3=7$ and prove that $s_4=9$.

2. For the equation $x^3+qx+r=0$, prove that

(i) $s_3=-3r$, (ii) $s_5=5qr$, (iii) $s_8=2q^4-8qr^2$, (iv) $6s_5=5s_2s_3$.

3. If $(x+a_1)(x+a_2)\dots(x+a_n) \equiv x^n + p_1x^{n-1} + \dots + p_n$, prove that $a_1^3 + a_2^3 + \dots + a_n^3 = p_1^3 - 3p_1p_2 + 3p_3$

4. If $s_r \equiv a^r + b^r + c^r + d^r$ and if $s_1 = s_2 = 0$, prove that

$$28s_{11} = 11s_4s_7.$$

5. For the equation $x^n + a_1x^{n-1} + \dots + a_n = 0$, prove that

(i) $a_3 = \frac{1}{6}(3s_1s_2 - 2s_3 - s_1^3)$

(ii) $s_4 = a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4$

B

6. For the equation $x^3 - 3x^2 + 4 = 0$, prove that $s_4 = 33$.

7. For the equation $x^5 - x^4 + 2x^3 - x^2 - x - 1 = 0$, prove that $s_4 = 9$.

8. For the equation $x^3 + a_1x^2 + a_3 = 0$, prove that $s_4 = 4a_1a_3 + a_1^4$.

9. If $s_r \equiv a^r + b^r + c^r$ and if $s_1 = 0$, prove that $s_6 = \frac{1}{3}s_3^2 + \frac{1}{2}s_2s_4$.

C

In Nos. 10-14, $\alpha, \beta, \gamma, \dots$ are the roots of $x^n + a_1x^{n-1} + \dots + a_n = 0$.

10. Express $\sum(\alpha + \beta - \gamma)^3$ in terms of the coefficients.

11. Prove that

$$\sum(x-\alpha)^p = nx^p - ps_1x^{p-1} + \binom{p}{2}s_2x^{p-2} - \dots + (-1)^p s_p$$

and deduce that $\sum(\alpha - \beta)^4 = ns_4 - 4s_1s_3 + 3s_2^2$.

Find a similar expression for $\sum(\alpha - \beta)^6$.

12. Assuming the formula $\dfrac{\partial a_{r+k}}{\partial s_r} = -\dfrac{a_k}{r}$ for $k = 1, 2, \dots, p-1$, prove that it is also true for $k = p$ by differentiating Newton's formula for s_{r+p} and using the formula for s_p.

13. Prove that $\sum\alpha^2\beta^2\gamma^2 = \frac{1}{6}(s_2^3 - 3s_2s_4 + 2s_6)$ and use this result to express $\sum\alpha^2\beta^2\gamma^2$ in terms of the coefficients.

14. Express $\sum(\alpha - \beta)^4$ in terms of the coefficients (see No. 11 and Example 13).

15. If $f(x) \equiv (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are all unequal, prove that

(i) $\sum_1^n \alpha_r^m / f'(\alpha_r) = 0$ if $0 < m < n-1$ and m is integral

(ii) the sum of the homogeneous products of k dimensions of the roots of $f(x) = 0$ is $\sum_1^n \alpha_r^{n+k-1}/f'(\alpha_r)$.

Cubic Equations. The classical method of solving a cubic equation was invented by Ferro and by Tartaglia, but was first published in 1545 by Cardan who had obtained it from Tartaglia after giving a solemn promise not to reveal it. In spite of these facts, it is generally called Cardan's method.

The solution of the general cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

may be made to depend upon that of a simpler cubic by the substitution $y=a_0x+a_1$. This gives

$$y^3+3Hy+G=0$$

where $H\equiv a_0a_2-a_1^2$; $G\equiv a_0^2a_3-3a_0a_1a_2+2a_1^3$.

We shall regard the equation as an equation of complex algebra Special interest is attached to the x-axal roots because of the application to real algebra. In any investigation into the nature of the roots it will be assumed that a_0, a_1, a_2, a_3 are x-axal numbers.

Tartaglia's Solution of $y^3+3Hy+G=0$.

The identity (see p. 258)

$$y^3-3pqy+p^3+q^3\equiv(y+p+q)(y+\omega p+\omega^2q)(y+\omega^2p+\omega q)$$

shows that if p, q are chosen so that

$$p^3+q^3=G \qquad pq=-H$$

the roots of the equation are $-p-q$, $-\omega p-\omega^2q$, $-\omega^2p-\omega q$. But p^3, q^3 are given as the roots of the quadratic equation $t^2-Gt-H^3=0$, and their cube roots p, q must then be chosen so that $pq=-H$. The roots of this quadratic are

$$\tfrac{1}{2}\{G\pm\sqrt{(G^2+4H^3)}\}.$$

If $G^2+4H^3>0$ p, q are x-axal and unequal. Hence the equation has one x-axal root, $-p-q$, and two conjugate roots. Thus if $y^3+3Hy+G=0$ is an equation of real algebra, it has just one root.

If $G^2+4H^3=0$ $p=q=\sqrt[3]{(\tfrac{1}{2}G)}$ and the roots are x-axal, two being equal. They are $-2p$, p, p. The corresponding equation of real algebra has then three roots $-2p$, p, p.

If $G^2+4H^3<0$ which implies $H<0$, the value of p^3 is of the form $A+iB$, $=r \operatorname{cis} \theta$, $(B \neq 0, \theta \neq n\pi)$, and $q^3=A-iB=r \operatorname{cis}(-\theta)$; also $pq=-H$. Thus p, q may be taken to be

$$(\sqrt[3]{r}) \operatorname{cis} \tfrac{1}{3}\theta, \quad (\sqrt[3]{r}) \operatorname{cis}(-\tfrac{1}{3}\theta),$$

where $\sqrt[3]{r}=\sqrt{(-H)}$, and the roots are

$$-p-q=-2(\sqrt[3]{r}) \cos \tfrac{1}{3}\theta$$

$$-\omega p-\omega^2 q=-(\sqrt[3]{r})\{\operatorname{cis}(\tfrac{1}{3}\theta+\tfrac{2}{3}\pi)+\operatorname{cis}(-\tfrac{1}{3}\theta-\tfrac{2}{3}\pi)\}$$

$$=-2(\sqrt[3]{r}) \cos(\tfrac{1}{3}\theta+\tfrac{2}{3}\pi)$$

and $$-\omega^2 p-\omega q=-2(\sqrt[3]{r}) \cos(\tfrac{1}{3}\theta-\tfrac{2}{3}\pi).$$

Hence the equation has three x-axal roots, and the corresponding equation of real algebra has three roots.

The Irreducible Case. When the equation has three roots in real algebra, these cannot be found by Tartaglia's method without using complex numbers. On the other hand when $G^2+4H^3>0$, the one root $-p-q$ can be found in real algebra without the introduction of ω, ω^2 by using the fact that $y+p+q$ is a factor of $y^3-3pqy+p^3+q^3$. In the sixteenth century the theory of complex numbers had not been invented and the case $G^2+4H^3<0$ came to be known as *irreducible.* It can however be solved by De Moivre's theorem as shown above or (without complex numbers) by the trigonometrical method illustrated in Example 15. See also *Advanced Trigonometry, p. 44.*

Whether there is one root or three, the method of Horner explained on p. 165 is usually more convenient in numerical examples.

Nature of the Roots of the General Cubic.

Since the roots of $a_0x^3+3a_1x^2+3a_2x+a_3=0$

and of $$y^3+3Hy+G=0$$

are connected by the relation $y=a_0x+a_1$, they are of the same nature, and this is decided by the sign of G^2+4H^3.

Putting $G^2+4H^3=a_0^2\Delta$, it follows from p. 307 that

$$\Delta=a_0^2a_3^2-6a_0a_1a_2a_3+4a_0a_2^3+4a_1^3a_3-3a_1^2a_2^2,$$

and then the sign of Δ decides the number of x-axal roots of either equation and therefore decides the number of roots in real algebra. For this reason Δ is called the *discriminant* of the cubic.

The critical case of equal roots is given by $\Delta=0$. See p. 307. This may also be proved by means of the theorem (p. 262) that $f(x)=0$ has a repeated root if $f(x)$ and $f'(x)$ have a common factor.

If α_1, α_2, α_3 are the roots of $a_0x^3+3a_1x^2+3a_2x+a_3=0$, with the notation of Example 10, p. 300, $a_0(\alpha_2-\alpha_3)=\beta-\gamma$, etc. and so by the result of that example,

$$(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_1)^2(\alpha_1-\alpha_2)^2=-27\Delta/a_0^4.$$

This explains from another standpoint why $\Delta=0$ is the condition for equal roots, and why $\Delta>0$ if and only if two roots are conjugate.

Example 14. Solve $3x^3-6x^2=2$ in real algebra.

The equation may be written

$$(3x-2)^3-12(3x-2)-34=0$$

or if $y=3x-2$, $\qquad y^3-12y-34=0,$

and this is the same as

$$y^3-3pqy+p^3+q^3\equiv(y+p+q)(y^2+p^2+q^2-yp-yq-pq)$$

if $p^3+q^3=-34$, $pq=4$.

Thus $\quad p^3, q^3$ are -2, -32.

Hence $\quad p=-\sqrt[3]{2}$

$\qquad\quad q=4/p=-2\sqrt[3]{4}$

But

$$y^2+p^2+q^2-yp-yq-pq=\tfrac{1}{2}\{(y-p)^2+(y-q)^2+(p-q)^2\}$$

and this is never zero in real algebra since $p\neq q$.

Therefore the equation has only one root

$$y=\sqrt[3]{2}+2\sqrt[3]{4}$$

and the given equation has only one root

$$x=\tfrac{1}{3}(2+\sqrt[3]{2}+2\sqrt[3]{4})$$

Example 15. Solve $x^3 - 6x - 4 = 0$ in complex algebra.

Tartaglia's Method. Choose p, q so that

$$p^3 + q^3 = -4, \ 3pq = 6.$$

p^3, q^3 are the roots of $t^2 + 4t + 8 = 0$, namely $-2 \pm 2i$.

Hence the roots of $x^3 - 6x - 4 = 0$ are

$$-p - q, \ -\omega p - \omega^2 q, \ -\omega^2 p - \omega q,$$

where $p = \sqrt[3]{(-2 + 2i)}$, $q = 2/p$.

But $\quad -2 + 2i = -2\sqrt{2} \operatorname{cis}(-\frac{1}{4}\pi)$,

$$\therefore \ p = -\sqrt{2} \operatorname{cis}(-\tfrac{1}{12}\pi), \ \ q = -\sqrt{2} \operatorname{cis}(\tfrac{1}{12}\pi),$$

and the roots are $\quad 2\sqrt{2} \cos(\frac{1}{12}\pi)$

$$\sqrt{2}\{\operatorname{cis}(\tfrac{7}{12}\pi) + \operatorname{cis}(-\tfrac{7}{12}\pi)\} = 2\sqrt{2} \cos(\tfrac{7}{12}\pi),$$

$$\sqrt{2}\{\operatorname{cis}(-\tfrac{3}{4}\pi) + \operatorname{cis}(\tfrac{3}{4}\pi)\} = 2\sqrt{2} \cos(\tfrac{3}{4}\pi) = -2$$

i.e. -2, $2\sqrt{2} \cos(\frac{1}{12}\pi)$, $2\sqrt{2} \cos(\frac{7}{12}\pi)$.

Trigonometrical Method. Put $x = k \cos \theta$,

then $\quad k^3 \cos^3\theta - 6k \cos \theta = 4$.

Choose k so that the left side is a multiple of $4 \cos^3\theta - 3 \cos \theta$ i.e. a multiple of $\cos 3\theta$

$$k^3 : 6k = 4 : 3 \ \text{ if } \ k = 2\sqrt{2}.$$

Then $\quad 16\sqrt{2} \cos^3\theta - 12\sqrt{2} \cos \theta = 4$,

$$\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta = \tfrac{1}{2}\sqrt{2} = \cos \tfrac{1}{4}\pi,$$

$$\therefore \ \theta = \tfrac{1}{12}\pi, \ \tfrac{3}{4}\pi, \text{ or } \tfrac{7}{12}\pi, \text{ or etc.}$$

$$\therefore \ x = 2\sqrt{2} \cos \tfrac{1}{12}\pi, \ -2, \text{ or } 2\sqrt{2} \cos \tfrac{7}{12}\pi$$

Algebraic Method. In this special case if the factor $x + 2$ is guessed, the equation is $(x + 2)(x^2 - 2x - 2) = 0$,

$$\text{and} \quad x = -2, \ 1 \pm \sqrt{3}.$$

EXERCISE XIIIe

[*In this exercise the algebra is complex unless otherwise stated*]

A

Solve the equations in Nos. 1-4 by Tartaglia's method.

1. $x^3 - 9x + 28 = 0$
2. $x^3 + 6x - 2 = 0$
3. $x^3 - 15x^2 - 33x + 847 = 0$
4. $x^3 + x^2 - 9x + 12 = 0$
5. Solve trigonometrically $x^3 - 27x - 27 = 0$

6. For what values of a, b can the equation $x^3 - ax = b$ be reduced to the form $\cos 3\theta = c$ where $|c| < 1$ by the substitution $x = k\cos\theta$?

7. Prove that the equation $x^3 + 3Hx + G = 0$ can be solved by expressing it in the form $\mu(x+\nu)^3 - \nu(x+\mu)^3 = 0$, where μ, ν are the roots of $Ht^2 - Gt - H^2 = 0$ and $G^2 + 4H^3 \neq 0$.

Apply this method to the equation $x^3 + 18x + 6 = 0$ in real algebra.

8. If α, β, γ are the roots of $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$, prove that $a_0^3(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) = -27G$.

9. Show that the equation $x^3 - ax = b$ can be solved by the substitution $x = k\operatorname{ch}\phi$ if $27b^2 > 4a^3 > 0$, and by the substitution $x = k\operatorname{sh}\phi$ if $a < 0$.

B

Solve the equations in Nos. 10-13 by Tartaglia's method.

10. $x^3 - 6x - 9 = 0$

11. $x^3 - 18x - 75 = 0$

12. $x^3 - 18x + 35 = 0$

13. $2x^3 + 6x^2 + 1 = 0$

14. Solve trigonometrically $x^3 + 3x^2 - 9x - 3 = 0$.

15. If α is a root of $x^3 = 3x + 1$, prove that the other two roots are $2 - \alpha^2$, $\alpha^2 - \alpha - 2$.

16. Solve the equation $x^3 - 15x^2 + 57x - 5 = 0$ in real algebra by substituting $y + 5$ for x and reducing the equation to the form $p(y-q)^3 = q(y-p)^3$.

17. Show that if the roots of $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots = 0$ are reduced by a suitable constant, the coefficients of the second and third terms in the transformed equation can both be made zero if $2na_2 = (n-1)a_1^2$.

18. If α, β, γ are the roots of $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$, prove that $\sum\{(a_0\alpha + a_1)^2(\beta - \gamma)^2\} = 18H^2/a_0^2$.

C

19. Solve $(x^3 - a^3 - b^3)^3 = 27a^3b^3x^3$

20. Prove that $\sqrt[3]{(2 + \frac{10}{9}\sqrt{3})} + \sqrt[3]{(2 - \frac{10}{9}\sqrt{3})} = 2$

21. Prove that the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ can be solved by expressing it in the form $\lambda(x-p)^3 + \mu(x-q)^3 = 0$, where p, q are the roots of $(a_0y + a_1)(a_2y + a_3) = (a_1y + a_2)^2$ and $\Delta \neq 0$.

Apply this method to the equation $x^3 - 3x^2 + 9x - 5 = 0$ in real algebra.

22. If $a_1^2 \neq a_2$, prove that the equation $x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ can be transformed by the substitution $x = y + m$ into the form $y^3 + 3py^2 + 3qy + pq = 0$ and hence that the solution may be written

$$(x - m + \sqrt{q})/(x - m - \sqrt{q}) = \{(p - \sqrt{q})/(p + \sqrt{q})\}^{\frac{1}{3}}$$

23. If $a_1^2 \neq a_2$, prove that the transformation $y(b - x) = a - x$ will reduce the equation $x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form $y^3 = c$ if a and b are the roots of

$$(a_1^2 - a_2)t^2 + (a_1a_2 - a_3)t + a_2^2 - a_1a_3 = 0.$$

What transformation gives this form if $a_1^2 = a_2$?

24. Reduce the equation $x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form $y^3 \pm 3y + m = 0$ by substituting $\lambda y + \mu$ for x, and show how to solve this equation by substituting $z \mp 1/z$ for y. Hence prove that the condition for equal roots is $G^2 + 4H^3 = 0$.

25. Show that if $\alpha_1, \alpha_2, \ldots$ are the roots of $f(x) = 0$ and if x is eliminated between $f(x) = 0$, $f(x + \sqrt{y}) = 0$, the roots of the resulting equation, excluding zero roots, are $(\alpha_1 - \alpha_2)^2$, $(\alpha_1 - \alpha_3)^2$, etc.

Use this method to find the equation whose roots are the squares of the differences of the roots of $x^3 + qx + r = 0$ and deduce the conditions for $x^3 + qx + r = 0$ to have

(i) a single non-repeated x-axal root, (ii) two equal roots.

Quartic Equations. The algebra is supposed complex, but in any investigation into the nature of the roots it is assumed that a_0, a_1, a_2, a_3, a_4 are x-axal numbers.

The solution of the general quartic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

is reduced by the substitution $y = a_0x + a_1$ to the simpler quartic

$$y^4 + 6Hy^2 + 4Gy + K = 0$$

where $\quad H = a_0a_2 - a_1^2 \quad G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$

and $\quad K = a_0^3a_4 + 6a_0a_1^2a_2 - 4a_0^2a_1a_3 - 3a_1^4$

$$= a_0^2(a_0a_4 - 4a_1a_3 + 3a_2^2) - 3(a_0a_2 - a_1^2)^2$$

$$= a_0^2I - 3H^2$$

where $\quad I = a_0a_4 - 4a_1a_3 + 3a_2^2.$

Methods of Solution. The solution of a quartic can be reduced to that of a cubic. Descartes' method depends on the determination of p, q, r such that (see Exercise XIIIf, No. 7)

$$y^4 + 6Hy^2 + 4Gy + K \equiv (y^2 - 2py + q)(y^2 + 2py + r).$$

No general method exists for the solution of equations of higher degree than 4 in terms of the literal coefficients.

Ferrari's Solution. This method like that of Descartes depends upon the expression of a quartic function in quadratic factors; but it is easily applicable to the general form of the equation and therefore saves the transformation by $y = a_0x + a_1$.

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

can be written

$$(a_0x^2 + 2a_1x + a_2 + 2\lambda)^2 = (2px + q)^2$$

if λ, p, q can be chosen so that

$$4a_1^2 + 2a_0(a_2 + 2\lambda) - 4p^2 = 6a_0a_2$$

$$4a_1(a_2 + 2\lambda) - 4pq = 4a_0a_3$$

$$(a_2 + 2\lambda)^2 - q^2 = a_0a_4.$$

$$p^2 = a_0\lambda + a_1^2 - a_0a_2 ; \quad pq = 2a_1\lambda + a_1a_2 - a_0a_3 ;$$

$$q^2 = (2\lambda + a_2)^2 - a_0a_4.$$

Thus

$$(4\lambda^2 + 4a_2\lambda + a_2^2 - a_0a_4)(a_0\lambda + a_1^2 - a_0a_2) = (2a_1\lambda + a_1a_2 - a_0a_3)^2,$$

or $$4a_0\lambda^3 - a_0\lambda(a_0a_4 - 4a_1a_3 + 3a_2^2) + (a_2^2 - a_0a_4)(a_1^2 - a_0a_2) - (a_1a_2 - a_0a_3)^2 = 0,$$

or $$4\lambda^3 - I\lambda + J = 0, \quad \text{where } J = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

This cubic for λ is called the *reducing cubic* of the quartic equation. Denoting its roots by λ_1, λ_2, λ_3, the corresponding values of p, $\equiv (a_0\lambda + a_1^2 - a_0a_2)^{\frac{1}{2}}$, by p_1, p_2, p_3, and the corresponding values of q, $\equiv (2a_1\lambda + a_1a_2 - a_0a_3)/p$, by q_1, q_2, q_3, the roots α, β, γ, δ of $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ are given by the two quadratic equations

$$a_0x^2 + 2(a_1 - p_1)x + (a_2 + 2\lambda_1 - q_1) = 0$$

$$a_0x^2 + 2(a_1 + p_1)x + (a_2 + 2\lambda_1 + q_1) = 0.$$

It is sufficient to take any one set of values λ_1, p_1, q_1.

The Roots of the reducing Cubic.

λ_1, λ_2, λ_3 can be expressed in terms of the differences $\alpha-\beta$, $\alpha-\gamma$, ... and in this way the nature of the roots of the quartic can be investigated.

If β, γ are the roots of the first quadratic,

$$\beta+\gamma=-2(a_1-p_1)/a_0 \qquad \alpha+\delta=-2(a_1+p_1)/a_0.$$

Similarly $\gamma+\alpha=-2(a_1-p_2)/a_0 \qquad \beta+\delta=-2(a_1+p_2)/a_0$

and $\alpha+\beta=-2(a_1-p_3)/a_0 \qquad \gamma+\delta=-2(a_1+p_3)/a_0.$

Hence
$$\beta-\gamma=2(p_3-p_2)/a_0 \qquad \alpha-\delta=2(p_3+p_2)/a_0$$
$$\gamma-\alpha=2(p_1-p_3)/a_0 \qquad \beta-\delta=2(p_1+p_3)/a_0$$
$$\alpha-\beta=2(p_2-p_1)/a_0 \qquad \gamma-\delta=2(p_2+p_1)/a_0$$

and
$$(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)=4(2p_1^2-p_2^2-p_3^2)/a_0^2$$
$$=4(2\lambda_1-\lambda_2-\lambda_3)/a_0$$
$$=12\lambda_1/a_0 \text{ since } \lambda_1+\lambda_2+\lambda_3=0.$$

Thus
$$\lambda_1=\frac{a_0}{12}\{(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)\}$$

similarly
$$\lambda_2=\frac{a_0}{12}\{(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)\}$$

and
$$\lambda_3=\frac{a_0}{12}\{(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)\}.$$

It is now possible to investigate the nature of the roots of the quartic. This depends as for the cubic on a discriminant Δ which is the *product of the squared differences* of the roots.

From what has been proved above

$$(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)=16(p_1^2-p_3^2)(p_2^2-p_1^2)/a_0^4.$$

$\therefore\ (\alpha-\beta)(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)=16(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)/a_0^2$ (see p. 313)

and by multiplication of three such results

$$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2$$
$$=16^3(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2(\lambda_1-\lambda_2)^2/a_0^6$$

which by Example 10, p. 300, $=-16^3.27\{(\tfrac{1}{4}J)^2+4(-\tfrac{1}{12}I)^3\}/a_0^6$
$$=256(I^3-27J^2)/a_0^6.$$

Put $\qquad I^3-27J^2=\Delta$

Then $\quad a_0^6(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2=256\Delta$

where Δ is called the *discriminant* of the quartic.

For an alternative method of evaluating the product of squared differences of the roots, see Exercise XIIIf, No. 8.

Nature of the Roots of the Quartic.

The roots of the general quartic are of the same nature as those of the quartic $y^4+6Hy^2+4Gy+(a_0^2I-3H^2)=0$ given by the substitution $y=a_0x+a_1$.

The necessary and sufficient condition for the roots to be all unequal is $\Delta\neq 0$. It also follows from the formula for Δ that

(i) if there are 4 unequal x-axal roots, $\Delta>0$,

(ii) if there are 2 unequal x-axal roots and 2 conjugate roots $(p\pm qi,\ q\neq 0)$, then $\Delta<0$,

(iii) if there are 2 pairs of conjugate roots, then $\Delta>0$.

It remains to distinguish between (i) and (iii), and to discuss the cases arising when $\Delta=0$.

If $G=0$, the quartic is a quadratic in y^2, and the reader will find no difficulty in investigating this special case. See Exercise XIIIf, No. 10. In what follows we therefore assume that $G\neq 0$.

The elimination of λ between

$$4\lambda^3-I\lambda+J=0 \qquad \mu\equiv p^2=a_0\lambda-H$$

gives a cubic in μ whose roots are p_1^2, p_2^2, p_3^2. This cubic is

$$4(\mu+H)^3-a_0^2I(\mu+H)+a_0^3J=0$$

or $$4\mu^3+12H\mu^2+(12H^2-a_0^2I)\mu+4H^3-a_0^2IH+a_0^3J=0$$

and is called *Euler's Cubic.*

It may be verified by direct multiplication or proved by the method of Exercise XIIIf, No. 26, that $G^2+4H^3\equiv a_0^2(IH-a_0J)$. Hence *Euler's* cubic may be written

$$4\mu^3+12H\mu^2+(12H^2-a_0^2I)\mu-G^2=0.$$

Assuming that $G\neq 0$, p_1^2, p_2^2, and p_3^2 are all different from zero. Then from relations like $4p_1=a_0\{(\beta+\gamma)-(\alpha+\delta)\}$ it follows that if α, β, γ, δ are all x-axal (case i) p_1^2, p_2^2, p_3^2 are all positive, but if α, β, γ, δ are 2 pairs of conjugates (case iii) one of p_1^2, p_2^2, p_3^2 is positive and the other two are negative.

Descartes' rule of signs applied to Euler's cubic shows that there are 3 positive roots only if $H<0$ and $12H^2-a_0^2I>0$. Hence if $\Delta>0$, $G\neq 0$, the quartic has 4 unequal x-axal roots if $H<0$ and $12H^2-a_0^2I>0$ and otherwise has two pairs of conjugate roots.

Equal Roots. If $\Delta=0$, there are at least two roots equal, and assuming $G\neq 0$ there can only be exactly two or exactly three equal roots. For if there were four, since the sum of the roots is zero, each would be zero and hence $G=0$. Also if there are two pairs of equal roots, they must be of the form $\alpha, \alpha, -\alpha, -\alpha$ and again $G=0$.

(iv) If only two roots are equal, say $\alpha=\beta$, these must be x-axal and the relations on p. 314 show that $\lambda_1=\lambda_2=-\frac{1}{2}\lambda_3\neq 0$. Hence $J\neq 0$, and from $\Delta=0$, $I\neq 0$.

As before, the other roots are x-axal if Euler's cubic has 3 positive roots, that is if $H<0$ and $12H^2>a_0{}^2I$; otherwise these remaining roots are conjugate.

(v) If exactly three roots are equal, they must be x-axal, and there must be a fourth x-axal root. Also $\lambda_1=\lambda_2=\lambda_3=0$. Thus $I=J=0$; these conditions imply $\Delta=0$.

Since all the possible systems of roots have been considered and the conditions are mutually exclusive, the converse statements are also true.

The roots of the original quartic for x are of the same nature as those of the quartic for y.

Reciprocal Equations. An equation which is unaltered when x is replaced by $1/x$ is called a *reciprocal* equation.

If $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$ is reciprocal, it is equivalent to

$$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0,$$

$$\therefore\ \frac{a_0}{a_n}=\frac{a_r}{a_{n-r}}=\frac{a_n}{a_0}$$

and so either (i) $a_r=a_{n-r}$ $(r=0, 1, 2, \ldots, n)$

or (ii) $a_r=-a_{n-r}$ $(r=0, 1, 2, \ldots, n)$.

(i) $a_r=a_{n-r}$. If the equation is of odd degree, it is satisfied by $x=-1$; and when the factor $x+1$ is removed, there remains a reciprocal equation of even degree of the form

$$a_0(x^{2m}+1)+a_1(x^{2m-1}+x)+\ldots+a_mx^m=0.$$

This may be written

$$a_0\left(x^m+\frac{1}{x^m}\right)+a_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\ldots+a_{m-1}\left(x+\frac{1}{x}\right)+a_m=0,$$

and the substitution of y for $x+1/x$ reduces it to an equation of degree m in y.

If the equation is of even degree, the same substitution is used without the removal of a factor.

(ii) $a_r=-a_{n-r}$. If the equation is of odd degree, it is satisfied by $x=1$; and when the factor $x-1$ is removed there remains a reciprocal equation of even degree of type (i).

If the equation is of even degree, it is satisfied by $x=1, x=-1$; and when the factor x^2-1 is removed there remains a reciprocal equation of even degree of type (i).

Example 16. Solve $12x^5-8x^4-45x^3+45x^2+8x-12=0$

Dividing by $x-1$,

$$12x^4+4x^3-41x^2+4x+12=0,$$

$$\therefore\ 12(x^2+1/x^2)+4(x+1/x)-41=0$$

or $$12(y^2-2)+4y-41=0 \quad \text{where } y=x+1/x.$$

Hence $$y=13/6 \text{ or } -5/2,$$

and $$x=\tfrac{2}{3},\ \tfrac{3}{2},\ -2,\ -\tfrac{1}{2}, \text{ or } 1.$$

A more general type can be solved by the substitution

$y=kx+1/x$. See Exercise XIIIf, No. 6.

EXERCISE XIIIf

[*Throughout this exercise the algebra is complex*]

A

Solve by Ferrari's method the equations in Nos. 1, 2.

1. $x^4-12x-5=0$

2. $x^4+4x^3-11x-4=0$

Solve the reciprocal equations in Nos. 3, 4.

3. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

4. $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$

5. If $a^2d = c^2$, prove that the product of two of the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ equals the product of the other two. Hence solve $x^4 - 2x^3 - 9x^2 + 6x + 9 = 0$

6. If $a : e = b^2 : d^2$, prove that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

can be solved by a substitution of the form $kx + 1/x = y$.

7. [*Descartes' method.*] If

$$y^4 + 6Hy^2 + 4Gy + I - 3H^2 \equiv (y^2 - 2py + q)(y^2 + 2py + r),$$

prove that $p^2 + H$ is a root of $4\lambda^3 - I\lambda + J = 0$, where

$$J \equiv IH - G^2 - 4H^3,$$

and that q, r are the roots of $t^2 - 2(2\lambda + H)t + I - 3H^2 = 0$.

8. Prove that if the product of the squared differences of the roots of $y^4 + 6Hy^2 + 4Gy + I - 3H^2 = 0$ is expressed in terms of the coefficients, G can only enter in even powers. Hence show by considerations of weight and order that the expression is of the form $lI^3 + mJ^2$, and deduce its value by considering (say) the equations $y^4 - 6y^2 = 0$, $y^4 - 1 = 0$.

9. Examine the nature of the roots of $y^4 + 4Gy + I = 0$.

10. For the general quartic, prove that if $G = 0$,

$$a_0^6\Delta = (a_0^2I - 12H^2)^2(a_0^2I - 3H^2),$$

and that

(i) if $12H^2 < a_0^2I$, there are two pairs of unequal conjugate roots,

(ii) if $12H^2 = a_0^2I$, there are two pairs of equal roots which are conjugate if $H > 0$, x-axal if $H < 0$, (all equal if $H = 0$),

(iii) if $3H^2 < a_0^2I < 12H^2$, there are four unequal roots which are conjugate pairs if $H > 0$ and x-axal if $H < 0$,

(iv) if $3H^2 = a_0^2I \neq 0$, there are two equal x-axal roots and two other roots which are conjugate or x-axal according as $H >$ or < 0,

(v) if $3H^2 > a_0^2I$, there are two unequal x-axal roots and two conjugate roots.

Under which headings are the cases $H = 0$, $I \neq 0$ and $H \neq 0$, $I = 0$ included?

B

Solve by Ferrari's method the equations in Nos. 11, 12.

11. $x^4 - 18x^2 + 16x - 3 = 0$ 12. $x^4 + 4x^3 - 6x^2 + 20x + 8 = 0$

Solve the reciprocal equations in Nos. 13, 14.

13. $2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2 = 0$ 14. $x^5 - 1 = 0$

15. Form the equation whose roots exceed by unity those of $x^5 - x^4 - 7x^3 + 2x^2 + 10x + 4 = 0$. Hence find x.

16. Find the condition that the sum of two roots of the equation $x^4 + mx^2 + nx + p = 0$ may be equal to the product of the other two. If this condition holds, show that the substitution $y = \frac{1}{2} + 1/x$, reduces the equation to one in which the sum of two roots is zero.

17. Show how to transform $x^4 + 2ax^3 + 4bx^2 + 8ax + 16 = 0$, so that it becomes a reciprocal equation.

18. Examine the nature of the roots of

$$y^4 + 6Hy^2 + 4Gy - 3H^2 = 0.$$

C

19. Solve $4(x^2 - x + 1)^3 - 27x^2(x - 1)^2 = 0$.

20. Express $4x^2 + 10x + 13$ and $x^2 - 8x - 2$ simultaneously in the form $A(x - \alpha)^2 + B(x - \beta)^2$ where α, β are the same for both.

21. Solve the equation $x^5 - 5x^3 + 5x + 1 = 0$ by the substitution $x = k \cos \theta$.

In Nos. 22-25, α, β, γ, δ are the roots of the quartic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

22. If λ is a root of the reducing cubic, prove that

$$a_0(\beta\gamma + \alpha\delta) = 4\lambda + 2a_2.$$

Hence find the equation whose roots are $\beta\gamma + \alpha\delta$, $\gamma\alpha + \beta\delta$, $\alpha\beta + \gamma\delta$.

23. Find the relation between α, β, γ, δ if a root of the reducing cubic $4\lambda^3 - I\lambda + J = 0$ is (i) H/a_0, (ii) $\surd(I/12)$.

24. Prove that $a_0^4\Sigma\{(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2\} = 192(3a_0J - 2HI)$.

25. Prove that $a_0^4\Sigma\{(\alpha - \beta)^4(\gamma - \delta)^2\} = -96(3a_0J + 4HI)$.

26. Deduce from the product of the roots of the reducing cubic that J/a_0^3 is a function of degree 6 of the differences between the roots of $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, and show that the same function of the differences between the roots of the quartic in y obtained by the substitution $y = a_0x + a_1$ is

$$\begin{vmatrix} 1 & 0 & H \\ 0 & H & G \\ H & G & a_0^2I - 3H^2 \end{vmatrix}.$$

Deduce that the value of this determinant is a_0^3J.

MISCELLANEOUS EXAMPLES

EXERCISE XIIIg

[*In this exercise the algebra is complex unless otherwise stated or implied*]

A

1. Find the conditions for the roots α, β, γ of

$$x^3 - ax^2 + bx - c = 0$$

to be in (i) A.P., (ii) G.P.

If the roots are not in A.P. and if $\alpha+\lambda$, $\beta+\lambda$, $\gamma+\lambda$ are in G.P., prove that λ is given by a cubic equation.

2. If a, b, c are positive and unequal and if $d>0$, prove that

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{d}{x}+1=0$$

has 4 roots in real algebra.

3. If α, β are the roots of $x^2+bx+c=0$ and γ, δ the roots of $x^2+(b+q)x+c+r=0$, express $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ in terms of b, c, q, r.

4. If $s \equiv x^2+2qx+r$ and $t \equiv x^2+c$, and if $s+\lambda t=0$ has equal roots in x and the two values of λ are equal, prove that in real algebra (i) if $c>0$, $s\equiv t$, (ii) if $c<0$, s and t have a common factor.

5. If the equation $x^n+a_1x^{n-1}+\ldots+a_n=0$ has n roots in real algebra, prove that $(n-1)a_1^2 \geqslant 2na_2$.

6. If α, β, γ are the roots of $x^3+3a_1x^2+3a_2x+a_3=0$, find the values of (i) $\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}\right)$ and (ii) $\sum\beta^3\gamma^3$.

7. If α, β, γ, δ are the roots of $x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$, prove that $\sum(\alpha-\beta)^2=48(a_1^2-a_2)$ and deduce that

$$\sum\alpha^2\beta^2(\gamma-\delta)^2=48(a_3^2-a_2a_4).$$

8. Find the condition that $3x^4+4px^3+q=0$ has no roots in real algebra.

9. If $q(q^7+r^3)<0$, prove that $x^7-7qx^4+21q^2x+15r=0$ has only one x-axal root. Is this a necessary condition ?

10. Use Sturm's theorem to find the number and signs of the roots of $2x^3-4x^2-2x+5=0$ in real algebra.

11. Prove that, if $m(m-1)\neq 0$, the equation $x^4+2x^3=2mx+m$ has exactly 2 roots in real algebra.

12. If $ax+cy+bz=p$, $cx+by+az=q$, $bx+ay+cz=r$, prove that $\sum(a^2-bc)\sum(x^2-yz)=\sum(p^2-qr)$.

13. Show that the equation $ax^5 - bx^3 + cx + d = 0$ in real algebra can be solved by the substitution $x = k\cos\theta$ if $b^2 = 5ac$ and $0 < a/b < 4c^2/(125d^2)$.

14. If $\beta_1, \beta_2, \beta_3, \beta_4$ are successive unrepeated roots of $f'(x) = 0$ in real algebra, where $f(x)$ is a polynomial such that $f(x) = 0$ has a root between β_1, β_2 and between β_3, β_4, prove that it also has a root between β_2, β_3.

15. If $\alpha, \beta, \gamma, \ldots$ are the roots of $x^n + a_1x^{n-1} + \ldots + a_n = 0$, prove that

(i) $\sum\alpha^2\beta^2\gamma\delta = a_2a_4 - 4a_1a_5 + 9a_6$

(ii) $\sum\alpha^3\beta\gamma = a_1a_4 + 2a_2a_3 - a_1^2a_3 - 5a_5$

16. If $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic and $\lambda_1, \lambda_2, \lambda_3$ of its reducing cubic, prove that $(\beta+\gamma)(\alpha+\delta) = 4(a_2 - \lambda_1)/a_0$, and deduce that $\sum(\beta+\gamma)^2(\alpha+\delta)^2 = 8(6a_2^2 + I)/a_0^2$

B

17. Find the condition for the roots of $ax^2 + 2bx + c = 0$ to exceed those of $px^2 + 2qx + r = 0$ by equal amounts.

18. Find the number and signs of the roots of $2x^6 - 8x^5 - 1 = 0$ in real algebra.

19. If $ax^3 + 3bx^2 + 3cx + d = 0$ has exactly two equal roots, prove that they are equal to $\frac{1}{2}(bc - ad)/(ac - b^2)$.

20. Prove that if $a < b < c$, the equation

$$\frac{1+ax}{x-a} + \frac{1+bx}{x-b} + \frac{1+cx}{x-c} + d = 0$$

in real algebra has three roots, and find their positions.

21. Solve by Tartaglia's method, $x^3 - 15x + 30 = 0$.

Use Sturm's theorem to find the number and signs of the roots of the equations in Nos. 22, 23, in real algebra :

22. $x^5 - x + 16 = 0$ 23. $x^4 + 4x^3 + 7x^2 + 6x - 4 = 0$

24. Prove that the equation $(x-a)^3(x-b)^3 + \lambda = 0$, $\lambda \neq 0$, has 2, 1, or no roots in real algebra according as $64\lambda <, =, > (a-b)^6$.

25. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, find the values of (i) $\sum(\alpha-\beta)^2$, (ii) $\sum\alpha^2\beta^2$, (iii) $\sum\alpha\beta(\gamma+\delta)^2$.

26. Find the condition that $x^4 + px^3 + qx^2 + rx + s = 0$ should have two roots whose sum is zero, and show how to solve the equation if this condition is satisfied.

C

27. If α, β, γ are the roots of $x(x^2-9)=k(x^2-1)$, express β, γ as rational functions of α.

28. Prove that the sum of the r^{th} powers of the roots of the equation, $x^n+x^{n-1}/1!+x^{n-2}/2!+\dots+1/n!=0$, is zero if

$$1<r<n+1.$$

29. Prove that in real algebra $x^5+5ax^3+5a^2x+b=0$ has exactly one root if $4a^5+b^2>0$ and has 5 roots if $4a^5+b^2<0$

30. Prove that in real algebra $x^3+x^2=\lambda x+1$ has three roots if $\lambda>1$ and only one root if $\lambda<1$.

31. If $u\equiv ax^2+2bx+c$, $v\equiv px^2+2qx+r$ and u, v are co-prime, and if α, β are the roots of $(ay+b)(qy+r)=(py+q)(by+c)$, prove that A, B, C, D can be found independent of x such that

$$u\equiv A(x-\alpha)^2+B(x-\beta)^2,\ v\equiv C(x-\alpha)^2+D(x-\beta)^2.$$

32. Prove that the equation

$$x^n+nx^{n-1}+\tfrac{1}{2}n(n-1)x^{n-2}+a_3x^{n-3}+\dots+a_n=0$$

has not more than $n-2$ roots in real algebra.

33. If n and p are odd positive integers and $n>p$, prove that in real algebra $x^n+ax^p+b=0$ has exactly one root if $a>0$ and has exactly three roots if $(pa/n)^n+\{pb/(n-p)\}^{n-p}<0$.

34. If $f(x)\equiv a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are all unequal, and if $\beta_1, \beta_2, \dots, \beta_{n-1}$ are the roots of $f'(x)=0$, prove that

$$f'(\alpha_1)f'(\alpha_2)\dots f'(\alpha_n)=a_0n^nf(\beta_1)f(\beta_2)\dots f(\beta_{n-1}).$$

35. If $\alpha, \beta, \gamma, \dots$ are the roots of $x^n+a_1x^{n-1}+\dots+a_n=0$, prove that $\sum\alpha^2\beta^2\gamma^2\delta^2=a_4^2-2a_3a_5+2a_2a_6-2a_1a_7+2a_8$

36. If $f(x)\equiv x^n+a_1x^{n-1}+\dots+a_n=0$ has k unequal positive roots (real algebra) prove that $xf'(x)+cf(x)=0$ has at least $k-1$ unequal positive roots.

37. If $f(x)\equiv x^n+a_1x^{n-1}+\dots+a_n=0$ has n unequal roots in real algebra, prove that $xf'(x)+(x+c^2)f(x)=0$ has $n+1$ roots.

38. If $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic, prove that

(i) $\sum(\beta-\gamma)^2(\gamma-\alpha)^2+\sum(\alpha-\beta)^2(\gamma-\delta)^2=8(a_0^2I+96H^2)/a_0^4$

(ii) $\sum(\beta+\gamma-\alpha-\delta)^2(\beta-\gamma)^2(\alpha-\delta)^2=192(3a_0J-2HI)/a_0^4$

39. If $f(x) \equiv x^n + a_1 x^{n-1} + \ldots + a_n$, find the condition that it may be possible to choose b, c so that the coefficients of x^{n-r}, x^{n-r-1} in $(x-b)(x-c)f(x)$ are both zero.

Hence prove that if $f(x) = 0$ has n roots in real algebra,

$$(a_r a_{r+3} - a_{r+1} a_{r+2})^2 < 4(a_{r+1}^2 - a_r a_{r+2})(a_{r+2}^2 - a_{r+1} a_{r+3}).$$

40. If $f(x)$, $g(x)$ are polynomials in real algebra such that

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)g(x)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are unequal, prove that $g(x) \equiv f^n(t)/n!$ where t lies between the greatest and least of $\alpha_1, \alpha_2, \ldots, \alpha_n, x$.

CHAPTER XIV

SEQUENCES

Sequences. If s_n is a one-valued function of n which is defined for all positive integral values of n, its values

$$s_1, s_2, s_3, \dots, s_n, \dots$$

are said to form a *sequence* (s_n).

Even if s_n is undefined for a finite number of positive integral values of n, (s_n) is still called a sequence.

Convergent Sequences. A sequence (s_n) is called *convergent* and is said to have *limit* l if $\lim_{n\to\infty} s_n = l$, and, in accordance with the definition on p. 55, this means that when an arbitrary positive number ϵ is given, there always exists a number m (which usually depends upon ϵ) such that $|s_n - l| < \epsilon$ for *every* integral value of n that is greater than m.

The sequence is also said to *converge to the limit* l.

We proceed to illustrate by examples the various ways in which a sequence may converge or fail to converge. There are two useful geometrical representations of a sequence : each term s_n may be represented by

(*a*) a point $x = s_n$ on an x-axis,

(*b*) a point with cartesian coordinates (n, s_n).

I $s_n = \dfrac{n-1}{2n}$

This defines the sequence $0, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{5}{12}, \dots$.

If ϵ is a given positive number, $|s_n - \frac{1}{2}| = \left| -\frac{1}{2n} \right| = \frac{1}{2n}$ and this is less than ϵ for all values of n greater than $1/(2\epsilon)$.

Therefore (s_n) is convergent and its limit is $\frac{1}{2}$.

No member of the sequence is equal to $\frac{1}{2}$. The limit of a sequence may or may not be a term of the sequence.

In this example the least value of the number m in the definition is a function of ϵ which can be calculated for any given value of ϵ. It is not usually possible to find a value of m that will serve for all values of ϵ.

In the representation (a) on the x-axis, there is an accumulation of points in the neighbourhood of $x=\frac{1}{2}$ and in this example they are all to the left of $x=\frac{1}{2}$.

In the representation (b), as n increases, the points steadily approach the line $y=\frac{1}{2}$ from below.

II $s_n = 1 + (-\frac{1}{2})^n$

This defines the sequence $\frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \frac{31}{32}, \ldots$.

Since $|s_n - 1| = 2^{-n} < \epsilon$ if $n > \left(\log \frac{1}{\epsilon}\right) / \log 2$, the sequence converges to the limit 1.

Here $s_n > 1$ if n is even and $s_n < 1$ if n is odd. Thus successive terms are alternately greater and less than the limit. Each term is actually nearer to the limit than any previous term, but this property is not necessary for convergence as will be illustrated in III. The reader should consider the representations (a) and (b) in this and the following examples.

III $s_n = \dfrac{\sin \frac{1}{2}n\pi}{n}$

This defines the sequence $1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, \ldots$

Since $|s_n| < \frac{1}{n} < \epsilon$, if $n > 1/\epsilon$, (s_n) converges to the limit 0.

The fact that $s_n = 0$ whenever n is even shows that the approach to the limit is not steady. In this example there is an unlimited number of terms of the sequence equal to the limit.

IV $s_n = \dfrac{\sin n}{n}$

Here also (s_n) converges to zero and the approach to the limit is not steady. There is no term of the sequence equal to the limit.

V $s_n = \dfrac{3n^2}{(n-1)(n-2)}$

If $n > 2$, $|s_n - 3| = (9n-6)/\{(n-1)(n-2)\} < 12(n-2)/(n-2)^2$ thus $|s_n - 3| < \epsilon$ if $n > 2 + 12/\epsilon$, and so (s_n) converges to 3.

Here the value $2 + 12/\epsilon$ taken for m is not the least value that will serve. To establish convergence it is enough to obtain *some* value of m for which the inequality is satisfied *whenever* $n > m$.

s_1 and s_2 do not exist, but s_n is defined for all other values of n; therefore the sequence (s_n) exists.

VI $s_n = \sec(\frac{1}{2}\pi\sqrt{n})$

This does not define a sequence as $\sec(\frac{1}{2}\pi\sqrt{n})$ is meaningless when n is the square of any odd number.

Divergent Sequences. The sequence (s_n) is called *divergent* and is said to *diverge to* $+\infty$ if when any number K whatever is assigned, there always exists a number m such that $s_n > K$ for *every* integral value of n that is greater than m. m usually depends upon K.

The divergence may be expressed by $\lim\limits_{n\to\infty} s_n = +\infty$ or by '$s_n \to +\infty$ when $n \to \infty$'.

Similarly the sequence (s_n) is called *divergent* and is said to *diverge to* $-\infty$ if when any number K whatever is assigned, there always exists a number m such that $s_n < K$ for *every* integral value of n that is greater than m. This may be expressed by $\lim\limits_{n\to\infty} s_n = -\infty$ or by '$s_n \to -\infty$ when $n \to \infty$'.

VII $s_n = \sqrt{n}$

Since $\sqrt{n} > K$ whenever $n > K^2$, (s_n) diverges to $+\infty$. Similarly $s_n = -\sqrt{n}$ defines a sequence which diverges to $-\infty$.

VIII $s_n = \{1 + (-1)^n\}\sqrt{n}$

This defines the sequence $0, 2\sqrt{2}, 0, 4, 0, 2\sqrt{6}, \ldots$

If n is even, $s_n = 2\sqrt{n}$; if n is odd, $s_n = 0$.

Although $s_n > K$ whenever n is an *even* number greater than $\frac{1}{4}K^2$, the sequence does not diverge to $+\infty$ because there is no value of m corresponding to a given positive K such that $s_n > K$ for *every* value of n that is greater than m. Nor is the sequence convergent.

The two phrases:

"for all sufficiently large values of n"

and "for values of n as large as we please"

are often used and must be carefully distinguished.

In the definitions of convergent and divergent sequences the inequalities $|s_n - l| < \epsilon$ and $s_n > K$ must be true for all sufficiently large values of n or more shortly *for all sufficiently large n.*

No sequence is defined in VI because $\sec(\frac{1}{2}\pi\sqrt{n})$ is meaningless *for values of n as large as we please.* Also in VIII $s_n > K$ for values of n as large as we please and $s_n < 1$ for values of n as large as we please.

Oscillatory Sequences. A sequence which neither converges, nor diverges to $+\infty$, nor diverges to $-\infty$, is called *oscillatory.*

If, in an oscillatory sequence, a constant C exists such that $|s_n| < C$ for all values of n for which s_n is defined, (s_n) is said to *oscillate finitely,* and otherwise it is said to *oscillate infinitely.*

The sequence in VIII oscillates infinitely.

IX $s_n = (-1)^{n+1}\left(1 + \frac{1}{n}\right)$

If n is odd, $|s_n - 1| = 1/n < \epsilon$ for all sufficiently large n, namely whenever $n > 1/\epsilon$.

If n is even, $|s_n + 1| = 1/n < \epsilon$ whenever $n > 1/\epsilon$.

But (s_n) does not converge to $+1$ or to -1. It is not enough that the inequality should hold for an unlimited number of values of n or for values of n as large as we please. No inequality $|s_n - l| < \epsilon$ is satisfied for all sufficiently large n and therefore (s_n) is not convergent.

Since $|s_n| < 3$ for all values of n, the sequence is not divergent but oscillates finitely.

Upper and Lower Bounds. If there exists a constant C such that s_n is never greater than C, the sequence (s_n) is said to be *bounded above*. It can then be proved that in the domain of real numbers (but not necessarily in that of rational numbers) a number M exists such that s_n is never greater than M, whereas if $M' < M$ there is some value of n for which $s_n > M'$. As a matter of fact M is the least value of C and it is called the *upper bound* of (s_n), but a proof would involve the definition of a real number.

In I (p. 324) $s_n < \frac{1}{2}$ for all values of n, but if $M' < \frac{1}{2}$, then $s_n > M'$ for some value of n (in fact here for an unlimited number of values of n). Therefore the upper bound M of (s_n) is $\frac{1}{2}$. Here no term of (s_n) is equal to M.

If $s_n = (n+1)/n$, $M = 2$ and $s_1 = M$. Here no other term of (s_n) is greater than $1\frac{1}{2}$.

Similarly if there exists a constant c such that s_n is never less than c, the sequence (s_n) is said to be *bounded below*, and the greatest value m of c is called the *lower bound* of (s_n). It is evident that if the upper bound of $(-s_n)$ is M, the lower bound of (s_n) is $-M$, and the lower bound may be so defined.

In I s_n is never negative and $s_1 = 0$. Therefore the lower bound m of (s_n) is 0.

If $s_n = (n+1)/n$, then $s_n > 1$ for all values of n, but if $m' > 1$, $s_n < m'$ for some value of n (in fact here for an unlimited number of values of n). Therefore $m = 1$. Here no term of (s_n) is equal to m.

Upper and Lower Limits. In IX, (s_n) is bounded above and $M = 2$. But *for all sufficiently large n*, s_n is less than any assigned number that exceeds 1, although this is not true of 1 itself. Therefore 1 is called the *upper limit* of (s_n) in that example. This is denoted by

$$\Lambda = \overline{\lim_{n\to\infty}}\, s_n = 1.$$

Also in IX, (s_n) is bounded below and $m = -1\frac{1}{2}$. But *for all sufficiently large n*, s_n is greater than any assigned number less than -1, although this is not true of -1 itself. Therefore -1 is called the *lower limit* of (s_n) in that example. This is denoted by

$$\lambda = \varliminf_{n\to\infty} s_n = -1.$$

If $\Lambda = \lambda = l$, (s_n) converges to the limit l.

If $\Lambda \neq \lambda$, (s_n) is oscillatory.

X $\quad s_n = \cos \frac{1}{3}n\pi + \dfrac{1}{n-1}$

s_1 does not exist. If $n > 1$ and p is a positive integer, then

for $n = 6p$, $s_n = 1 + 1/(n-1)$; for $n = 6p \pm 1$, $s_n = \frac{1}{2} + 1/(n-1)$;

for $n = 6p \pm 2$, $s_n = -\frac{1}{2} + 1/(n-1)$;

for $n = 6p + 3$, $s_n = -1 + 1/(n-1)$.

Hence (s_n) is represented by points on an x-axis which accumulate near $x = \pm 1$, $x = \pm\frac{1}{2}$. It happens that $s_2 = \frac{1}{2}$, $s_3 = -\frac{1}{2}$, but no s_n is equal to ± 1.

Also $M = 1\frac{1}{3}$, $m = -1$, and $\Lambda = 1$, $\lambda = -1$.

XI $\quad s_n = (\sqrt{n}) \cos \frac{1}{2}n\pi + \sin \frac{1}{3}n\pi - \dfrac{1}{n}$

If n is even, $s_n = (-1)^{n/2}\sqrt{n} + \alpha$, where $|\alpha| < 1$. Therefore for all sufficiently large even values of n, $s_n > K$ if $\frac{1}{2}n$ is even and $s_n < K$ if $\frac{1}{2}n$ is odd.

If n is odd, $s_n = \sin \frac{1}{3}n\pi - \dfrac{1}{n}$ and the method used in X shows that the representative points on an x-axis accumulate near $x = \pm 1$, $x = \pm\frac{1}{2}$ which are called *points of accumulation* of (s_n).

This sequence oscillates infinitely and we may write

$$\varlimsup_{n\to\infty} s_n = +\infty, \quad \varliminf_{n\to\infty} s_n = -\infty.$$

For the sequence in VIII, p. 326, which also oscillates infinitely,

$$\varlimsup_{n\to\infty} s_n = +\infty, \text{ but } \varliminf_{n\to\infty} s_n = 0.$$

Monotone Sequences. If $s_{n+1} > s_n$ for all values of n, the sequence (s_n) is called *monotone increasing* or s_n is said to *increase steadily*.

For example 2, 4, 8, 16, ... and $\frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{4}{5}, \frac{5}{6}, \ldots$ are monotone increasing sequences.

When the terms s_n are all positive the condition $s_{n+1} > s_n$ may be replaced by $\frac{s_{n+1}}{s_n} > 1$.

For example if $a > 0$ and $s_n = \frac{(a+1)(a+2)\ldots(a+n)}{1.2\ldots n}$ the sequence (s_n) is monotone increasing,

for $\quad s_{n+1}/s_n = (a+n+1)/(n+1) > 1.$

Similarly if $s_{n+1} < s_n$ for all values of n, the sequence (s_n) is called *monotone decreasing* or s_n is said to *decrease steadily*.

For example $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotone decreasing sequence.

We shall continue to assume the theorems stated on p. 62 that a monotone increasing sequence (s_n) for which $s_n < C$ for all values of n, converges to a limit l such that $l \leqslant C$, and that a monotone decreasing sequence for which $s_n > c$ for all values of n, converges to a limit l such that $l \geqslant c$. These results cannot be proved without first establishing a theory of real numbers. They are not in fact true within the domain of rational numbers.

Example 1. If

$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n \quad \text{and} \quad t_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1} - \log n,$$

prove that (s_n) is monotone decreasing and that (t_n) is monotone increasing and that each converges to the same limit γ where $0{\cdot}3 < \gamma < 1$.

$$s_{n+1} - s_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right),$$

similarly $\qquad t_{n+1} - t_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right);$

but from p. 108, writing $\frac{1}{n}$ for u in equation (8),

$$\frac{1}{n+1} < \log\left(1 + \frac{1}{n}\right) < \frac{1}{n},$$

$$\therefore\ s_{n+1} - s_n < 0 \quad \text{and} \quad t_{n+1} - t_n > 0;$$

$\therefore$ (s_n) is monotone decreasing and (t_n) is monotone increasing.

Also since $\frac{1}{n} > \log(n+1) - \log n$, $\sum_1^n \frac{1}{r} > \log(n+1) > \log n$,

$\therefore$ s_n is positive, and since (s_n) is monotone decreasing, it converges to a limit γ where $\gamma \geqslant 0$. But $s_1 = 1$, $\therefore$ $\gamma < 1$.

Also $s_n - t_n = \frac{1}{n}$, $\therefore$ $\lim(s_n - t_n) = 0$, $\therefore$ (t_n) converges to the same limit γ.

But $t_2 = 1 - \log 2 > 0{\cdot}3$ (see p. 112) $\therefore$ $\gamma > 0{\cdot}3$.

The common limit of these two sequences is called *Euler's Constant.* Its value is ·5772157 ... and it is denoted by γ.

Example 2. Prove that the sequence (s_n) defined by

$$s_{n+1} = \frac{6(1+s_n)}{7+s_n}, \quad s_1 = c > 0$$

is monotone.

$$s_{n+1} - s_n = \frac{6(1+s_n)}{7+s_n} - s_n = \frac{6 - s_n - s_n^2}{7+s_n} = \frac{(2-s_n)(3+s_n)}{7+s_n}$$

and $$2 - s_{n+1} = 2 - \frac{6(1+s_n)}{7+s_n} = \frac{4(2-s_n)}{7+s_n}.$$

Hence if $s_n < 2$, $s_n < s_{n+1} < 2$, and so if $c < 2$,

$$s_1 < s_2 < s_3 < \ldots < 2$$

i.e. the sequence is monotone increasing.

Similarly if $c > 2$ it follows that $s_1 > s_2 > s_3 > \ldots > 2$ and so the sequence is monotone decreasing.

If $c = 2$, $s_n = 2$.

EXERCISE XIVa

A

1. If $s_n = 2 + (-\frac{1}{2})^n$, prove that $|s_n - 2| < {\cdot}001$ if $n > 9$ and find the least integer m such that $|s_n - 2| < 10^{-6}$ whenever $n > m$.

2. If $s_n = (2n+5)/(6n-11)$, prove that (s_n) is convergent and find the least integer m such that $|s_n - \frac{1}{3}| < {\cdot}001$ whenever $n > m$.

3. If $s_n = \sqrt[3]{(n-2)}$, prove that (s_n) is divergent.

4. If $s_n = (-1)^n(2n-1)/n$, prove that (s_n) oscillates finitely.

Find the nature of the sequences in Nos. 5-10 and their limits or upper and lower limits when these exist.

5. $(5n+2)/(n-1)$ 6. $(-1)^n$ 7. $(n^2-1)/n$

8. $n+(-1)^n n^2$ 9. $\tan^{-1} n$ 10. $\cos \frac{1}{2}n\pi + 1/n$

11. Give the upper and lower bounds and the points of accumulation on an x-axis in Nos. 5, 9, 10.

12. Prove that if $s_{n+1} = 2(1+s_n)/(3+s_n)$ and $s_1 > 0$, the sequence (s_n) is monotone.

B

13. If $s_n = (3n^2+5)/(4n^2-7)$, prove that (s_n) is convergent and find the least value of m if $|s_n - \frac{3}{4}| < \epsilon$ whenever $n > m$.

Answer the same questions as in Nos. 5-10 for Nos. 14-24.

14. $(\frac{2}{3})^n$ 15. $\frac{1}{2}\{1+(-1)^n\}$ 16. $\sqrt{(n^2-n)}$

17. $\tan \frac{1}{4}n\pi$ 18. $(n^2-1)/(n^2-2n)$ 19. $(-1)^n(1+1/n)$

C

20. $n^2+(-1)^n 2n$ 21. $(\sin n)/\sqrt{n}$ 22. $(\sqrt{n}) \cos n\pi$

23. $(\cos \frac{1}{2}n\pi + n \sin \frac{1}{2}n\pi)^{-1}$ 24. $(\sqrt{n}) \sin^2 \frac{1}{2}n\pi + \cos \frac{1}{2}n\pi - n^{-1}$

25. Give the upper and lower bounds and the points of accumulation on an x-axis in Nos. 20, 21, 23, 24

26. Give the lower limit of the sequence in No. 24.

27. If $s_{n+1} = 12/(1+s_n)$ and $0 < s_1 < 3$, show that $s_1, s_3, s_5, \ldots$ and $s_2, s_4, s_6, \ldots$ are monotone sequences which are increasing and decreasing respectively. Examine also the case $s_1 > 3$.

28. Prove that the sequence (s_n) given by $s_{n+1} = \sqrt{(6+s_n)}$, $s_1 > 0$, is monotone. Examine the cases $s_1 < 3$, $s_1 > 3$.

Some important Limits.

(i) If $-1 < x < +1$, $\lim_{n\to\infty} x^n = 0.$

(ii) For all values of x, $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

(iii) $\lim_{n\to\infty} \frac{\log n}{n} = 0.$

(iv) If $\alpha > 0$, $\lim_{n\to\infty} \frac{\log n}{n^\alpha} = 0.$

The first two were given on pp. 56, 76. Direct elementary proofs will be found in *Durell and Robson : Advanced Trigonometry, p. 78* ; and a proof of (iii) is given in the same book on p. 68. It is easy to deduce (iv) from (iii).

(v) *If the sequence* (s_n) *converges to the limit zero, then the sequence* (t_n) *defined by* $t_n = (s_1 + s_2 + \ldots + s_n)/n$ *also converges to zero.*

First suppose then $s_n \geqslant 0$ for all values of n.

Since (s_n) converges to zero, if ϵ is any given positive number, m exists such that $s_n < \frac{1}{2}\epsilon$ whenever $n > m$.

Thus
$$t_n \equiv (s_1 + s_2 + \ldots + s_m)/n + (s_{m+1} + \ldots + s_n)/n$$
$$< (s_1 + s_2 + \ldots + s_m)/n + \tfrac{1}{2}(n-m)\epsilon/n$$
$$< (s_1 + s_2 + \ldots + s_m)/n + \tfrac{1}{2}\epsilon$$

Since m is fixed, n_1 can be chosen, greater than m, so that
$$(s_1 + s_2 + \ldots + s_m)/n_1 < \tfrac{1}{2}\epsilon,$$
and then $\quad t_n < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ whenever $n > n_1 > m$.

Thus (t_n) converges to zero.

Now suppose that s_n may be positive or negative.

Since (s_n) converges to zero, so also does $(|s_n|)$.

But $\quad |t_n| = |(s_1 + s_2 + \ldots + s_n)/n| \leqslant (|s_1| + |s_2| + \ldots + |s_n|)/n$;

hence $(|t_n|)$ converges to zero, and $\therefore$ (t_n) also does so.

(vi) *If the sequence* (u_n) *converges to the limit* l, *then the sequence* (v_n) *defined by* $v_n = (u_1 + u_2 + \ldots + u_n)/n$ *also converges to* l.

For put $u_n = l + s_n$; then (s_n) converges to zero and therefore by (v), $(s_1 + s_2 + \ldots + s_n)/n \to 0$ when $n \to \infty$.

But $v_n = l + (s_1 + s_2 + \ldots + s_n)/n$, $\therefore$ (v_n) converges to l.

It should be noted that the converses of (v) and (vi) are not true. The convergence of (v_n) in (vi) does not imply that of (u_n). See Exercise XIVb, Nos. 1, 18.

Example 3. If $|a|<1$, prove that the sequence (s_n) defined by $s_n = na^n$ is convergent and that its limit is zero.

(i) Suppose $0<a<1$ and put $a=1-x$.

Then $0<x<1$ and $(1-x)(1+x)=1-x^2<1$, $\therefore\ a<1/(1+x)$.

Thus $$\frac{s_{n+1}}{s_n}=\frac{(n+1)a}{n}<\frac{1+1/n}{1+x}<1 \text{ when } n>\frac{1}{x}.$$

Hence, except possibly for a finite number of terms at the beginning, (s_n) is monotone decreasing. But $s_n>0$ for all values of n, $\therefore$ (s_n) is convergent.

Since $(1+x)^n=1+nx+\frac{1}{2}n(n-1)x^2+\ldots>\frac{1}{2}n(n-1)x^2$,

$$na^n<\frac{n}{(1+x)^n}<\frac{2}{(n-1)x^2}<\epsilon \text{ if } n>1+\frac{2}{\epsilon(1-a)^2}.$$

Therefore $na^n \to 0$ when $n\to\infty$.

This is also true when $a=0$.

(ii) Now suppose $-1<a<0$.

By (i), $|na^n|\to 0$ when $n\to\infty$, hence also $na^n\to 0$.

Alternatively from p. 69, the series $\sum n|a|^n$ is convergent if $|a|<1$, $a\neq 0$, since $\lim\,(s_n/s_{n+1})=1/|a|>1$; and therefore by the theorem on p. 64, $|s_n|\to 0$ when $n\to\infty$.

Example 4. Prove that the sequence $(\sqrt[n]{n})$ converges to the limit 1.

Put $\sqrt[n]{n}=1+x$. Then $(1+x)^n=n$, $\therefore\ x>0$.

Hence, as in Example 3, $n\equiv(1+x)^n>\frac{1}{2}n(n-1)x^2$.

Thus $$|\sqrt[n]{n}-1|\equiv x<\sqrt{\{2/(n-1)\}}<\epsilon \text{ if } n>1+2/\epsilon^2.$$

$$\therefore\ \sqrt[n]{n}\to 1 \text{ when } n\to\infty.$$

Alternatively from p. 108

$$\log\sqrt[n]{n}=\frac{1}{n}\log n=\frac{2}{n}\log\sqrt{n}<\frac{2}{\sqrt{n}}$$

and so $\log\sqrt[n]{n}<\epsilon$ when $n>4/\epsilon^2$.

Hence $\log\sqrt[n]{n}\to 0$ when $n\to\infty$

and, assuming the continuity of the logarithm, $\sqrt[n]{n}\to 1$ when $n\to\infty$.

Example 5. Prove that if $a>0$ the sequence $(\sqrt[n]{a})$ converges to the limit 1.

(i) Suppose $a>1$. Then if $n>a$, $1<\sqrt[n]{a}<\sqrt[n]{n}$.

But by Example 4, $\sqrt[n]{n}\to 1$ when $n\to\infty$; hence $\sqrt[n]{a}\to 1$.

(ii) Suppose $0<a<1$. Put $\sqrt[n]{a}=x$, then $a=x^n$ and $0<x<1$.

Thus $$(1-x^n)/(1-x)=1+x+\ldots+x^{n-1}>nx^n$$

and $$1-\sqrt[n]{a}\equiv 1-x<\frac{1-x^n}{nx^n}=\frac{1-a}{na},$$

$$\therefore\ |1-\sqrt[n]{a}|<\epsilon \text{ whenever } n>(1-a)/(a\epsilon),$$

$$\therefore\ \sqrt[n]{a}\to 1 \text{ when } n\to\infty.$$

Alternatively (ii) may be deduced from (i) by writing $1/a$ for a.

Example 6. Prove that the sequence (s_n) is convergent if

$$s_n=\frac{3.5.7\ldots 2n+1}{2.5.8\ldots 3n-1}.$$

$\dfrac{2n+1}{3n-1}<\dfrac{2\frac{1}{4}}{2\frac{1}{2}}=\dfrac{9}{10}$ if $20n+10<27n-9$ and therefore if $n>3$; hence

$$s_n<\frac{3}{2}\left(\frac{9}{10}\right)^{n-2} \quad \text{if } n>3.$$

But $\left(\dfrac{9}{10}\right)^{n-2}\to 0$ when $n\to\infty$; therefore $s_n\to 0$ when $n\to\infty$.

Note. In this proof instead of $2\frac{1}{4}$, $2\frac{1}{2}$, any two numbers α, β, such that $2<\alpha<\beta<3$ could be used.

Example 7. Prove that the sequence (s_n) is divergent if

$$s_n=\frac{2.5.8\ldots 3n-1}{1.4.7\ldots 3n-2}.$$

Since $\log t>(t-1)/t$ if $t>0$ and $t\neq 1$, by p. 108,

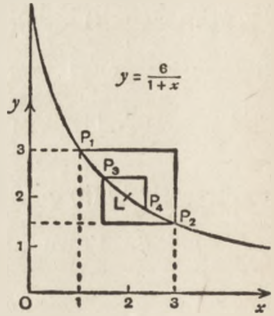
$$\log\frac{3n-1}{3n-2}>\frac{1}{3n-1}>\frac{1}{3n}.$$

Hence $$\log s_n>\tfrac{1}{3}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}\right).$$

But $1+\frac{1}{2}+\frac{1}{3}+\ldots+\dfrac{1}{n}\to\infty$ when $n\to\infty$, by p. 64; therefore $\log s_n\to\infty$ and $s_n\to\infty$ when $n\to\infty$.

The convergence of a sequence can sometimes be established by dividing the sequence into two sub-sequences each of which is monotone. This is illustrated by Example 8.

Example 8. If $s_1 = 1$ and $s_{n+1} = 6/(1+s_n)$, prove that (s_n) is a convergent sequence with limit 2.



If P_r denotes the point (s_r, s_{r+1}), it lies on the curve $y = 6/(1+x)$ for all values of r. The positions of P_1, P_2, P_3, P_4 are shown in the diagram and the positions of P_5, P_6, ... may be added in succession; these give some idea of the nature of the sequence.

If it is assumed that $s_n \to l$, then $s_{n+1} \to l$ and

$s_{n+1} = 6/(1+s_n)$ gives $l = 6/(1+l)$, $l^2 + l = 6$, $(l-2)(l+3) = 0$; but as $s_n > 0$, $l > 0$, therefore $l = 2$.

It is essential however to prove the existence of the limit.

If $s_n < 2$, $s_{n+1} = 6/(1+s_n) > 2$; if $s_n > 2$, $s_{n+1} < 2$; but $s_1 < 2$, hence $s_n < 2$ if n is odd, and $s_n > 2$ if n is even.

Also $$s_{n+2} - s_n = \frac{6}{1+s_{n+1}} - s_n = \frac{6(1+s_n)}{(1+s_n)+6} - s_n = \frac{(3+s_n)(2-s_n)}{7+s_n}.$$

Thus $s_{n+2} > s_n$ if n is odd, and $s_{n+2} < s_n$ if n is even.

Hence $s_1, s_3, s_5, \ldots$ is a monotone increasing sequence with positive terms all less than 2, and $s_2, s_4, s_6, \ldots$ is a monotone decreasing sequence with all terms greater than 2. Thus s_{2n-1} tends to a limit b, and s_{2n} tends to a limit c, where $0 < b < 2 < c$.

Since $s_{2n-1} \to b$, $s_{2n+1} - s_{2n-1} \to 0$, and from the expression found for $s_{n+2} - s_n$, $(3+b)(2-b) = 0$. But $b > 0$, $\therefore b = 2$.

Similarly by taking even values of n it follows that $c = 2$. Thus (s_n) converges to the limit 2.

Note. The result is true for all positive values of s_1. But if $s_1 > 2$, $s_1, s_3, s_5, \ldots$ is monotone decreasing and $s_2, s_4, s_6, \ldots$ is monotone increasing. The point $L(2, 2)$ in the diagram is the point to which P_r tends when $r \to \infty$.

EXERCISE XIVb

A

1. If $s_n = (-1)^n$ and $t_n = (s_1 + s_2 + \ldots + s_n)/n$, prove that (t_n) converges although (s_n) does not.

2. If $s_n = \sqrt[n]{n^3}$, prove that (s_n) converges to 1.

3. If $s_n = n^2 a^n$, $0 < a < 1$, prove that (s_n) converges to 0.

4. Prove directly that $(\log n)/\sqrt{n} \to 0$ when $n \to \infty$.

5. Prove that $(\log n)^{1/n} \to 1$ when $n \to \infty$

6. Prove that $\dfrac{4.7.10 \ldots 3n+1}{3.8.13 \ldots 5n-2} \to 0$ when $n \to \infty$.

7. Prove that the sequence (s_n) defined by $s_n = \dfrac{2.4.6 \ldots 2n}{1.3.5 \ldots 2n-1}$ is divergent.

8. If $s_1 > 0$ and $s_{n+1} = 2/(1 + s_n)$, prove that (s_n) converges to unity.

9. If $s_1 = 1$, $s_2 = 3$, $s_{n+2} = \frac{1}{2}(s_{n+1} + s_n)$, prove that the subsequence $s_1, s_3, s_5, \ldots$ is monotone increasing and that $s_2, s_4, s_6, \ldots$ is monotone decreasing, and that (s_n) converges to $2\frac{1}{3}$.

B

10. If $s_n = \sqrt[n]{a}$, $a > 0$, prove that (s_n) is monotone.

11. If (s_n) converges to zero and $t_n = 2^{s_n}$, prove that (t_n) converges to unity.

12. Prove that the sequence (s_n) defined by $s_n = \dfrac{3.7.11 \ldots 4n-1}{4.7.10 \ldots 3n+1}$ is divergent.

13. Prove that $\dfrac{1.3.5 \ldots 2n-1}{4.6.8 \ldots 2n+2} \to 0$ when $n \to \infty$.

14. If $s_n = n^k a^{-n}$, $a > 1$, $k > 0$, prove that (s_n) converges to zero.

15. If $ns_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n$, prove that (s_n) converges to zero.

16. If $s_n = (1 - x^n)/(1 - x)$, $0 < x < 1$, verify from first principles that $\lim s_n = \lim \{(s_1 + s_2 + \ldots + s_n)/n\}$

17. If $s_1 = 1$, $s_2 = 6$, $6s_{n+2} = 5s_{n+1} - s_n$, prove that (s_n) converges to zero.

C

18. If $s_n = n(-1)^{n-1}$ and $t_n = (s_1 + s_2 + \dots + s_n)/n$ and

$$v_n = (t_1 + t_2 + \dots + t_n)/n,$$

find the nature of the sequences (t_n) and (v_n).

19. If $s_n = n^{2n}(1+n^2)^{-n}$, prove that (s_n) converges to 1.

20. Evaluate $\lim\limits_{n\to\infty} (\sqrt{2} + \sqrt[3]{3} + \dots + \sqrt[n]{n})/n$

21. If $\lim\limits_{n\to\infty} \{f(n) - f(n-1)\} = l$, prove that $\lim\limits_{n\to\infty} \dfrac{f(n)}{n} = l$.

22. If $s_n = m(m+1)\dots(m+n-1)/n!$ and $m > 1$, prove that (s_n) is divergent.

23. If $m > -1$, prove that $m(m-1)\dots(m-n+1)/(n!) \to 0$ when $n \to \infty$.

24. If $s_{n+2} = (n+3)s_{n+1} - (n+1)s_n$, prove that

$$\lim_{n\to\infty} s_n/(n!) = (e-2)s_2 - (2e-5)s_1$$

25. If $s_n = u_1 + u_2 + \dots + u_n$ and $u_{n+2} + 2au_{n+1} + bu_n = 0$, prove that (s_n) is convergent if $-1 < b < a^2 < 1$ and $(1+b)^2 > 4a^2$. Examine the case $b = a^2 < 1$.

Infinite Series. The discussion of the convergence of

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is equivalent to that of the convergence of the sequence (A_n), where $A_n = a_1 + a_2 + a_3 + \dots + a_n$.

$\sum a_n$ converges if $\lim\limits_{n\to\infty} A_n$ exists.

It is convenient to use A_n, B_n, ... in this way to denote

$$a_1 + a_2 + \dots + a_n,\ b_1 + b_2 + \dots + b_n,\ \dots$$

and also to use A, B, ... for $\lim A_n$, $\lim B_n$, ... if these limits exist.

The usual tests for convergence of simple series were explained in Ch. IV, pp. 57-76. It was proved on p. 64 that if $\sum a_n$ converges, then $\lim a_n = 0$ and this is true whether the terms are all positive or not. But this test, though necessary, is not sufficient for convergence.

We shall now prove a stronger result of the same kind, applicable however only to series of positive terms. On pp. 342-346 we shall give some further tests of convergence.

Abel's Theorem. *If $\sum a_n$ is a convergent series of* **positive** *terms and if (a_n) is* **monotone decreasing,** *then* $\lim_{n\to\infty}\{na_n\}=0$.

By hypothesis A_n is monotone increasing and tends to a limit A. Hence for any given positive value of ϵ, k exists such that

$$A-\epsilon<A_n\leqslant A \text{ if } n>k,$$

$$\therefore\ A_n-A_k<\epsilon \text{ if } n>k,$$

i.e. $$a_{k+1}+a_{k+2}+\ldots+a_n<\epsilon.$$

$\therefore$ since $$a_{k+1}\geqslant a_{k+2}\geqslant a_{k+3}\geqslant\ldots\geqslant a_n>0,$$

$$(n-k)a_n<\epsilon,$$

$$\therefore\ 0<\tfrac{1}{2}na_n<\epsilon \text{ when } n\geqslant 2k.$$

Hence $na_n\to 0$ when $n\to\infty$.

Exercise XIVc, No. 9, shows that if (a_n) is not monotone decreasing, $\sum a_n$ may converge even when $\lim(na_n)$ is not zero.

For an illustration of the use of Abel's Theorem see Example 9.

The example $a_n=1/(n\log n)$ shows that the *converse theorem* that if (a_n) is a monotone decreasing sequence of positive terms and $\lim\{na_n\}=0$, then $\sum a_n$ is convergent, is *not true.* For $\sum 1/(n\log n)$ is divergent, by p. 343.

Example 9. If $a_n=n^{-1-1/n}$, prove that $\sum a_n$ is divergent.

First method. Since
$$\frac{d}{dx}\log x^{1+1/x}=\frac{d}{dx}\left\{\left(1+\frac{1}{x}\right)\log x\right\}$$
$$=\left(1+\frac{1}{x}\right)\frac{1}{x}-\frac{1}{x^2}\log x$$
$$=(x+1-\log x)/x^2>0, \text{ p. } 108,$$

it follows that $\log x^{1+1/x}$ increases with x and hence $x^{1+1/x}$ increases with x. Thus (a_n) is monotone decreasing.

Also $\lim\{na_n\}=\lim n^{-1/n}=1$, by Example 4, p. 334.

Hence by Abel's Theorem, $\sum a_n$ cannot converge.

But a_n is positive, therefore $\sum a_n$ diverges.

Second method. Since $\log n<n$, $n^{1/n}<e$,

$$\text{thus } a_n\equiv\frac{1}{n\,.\,n^{1/n}}>\frac{1}{ne}.$$

But $\sum(1/n)$ is divergent, therefore $\sum a_n$ is divergent.

Example 10. Examine the convergence of $\sum a_n$ when

$$a_n = n^k \left\{ \frac{1}{\sqrt{(n-1)}} - \frac{1}{\sqrt{n}} \right\}, \quad n > 1.$$

$$\frac{1}{\sqrt{(n-1)}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n} - \sqrt{(n-1)}}{\sqrt{(n^2-n)}} = \frac{1}{\sqrt{(n^2-n)}\{\sqrt{n} + \sqrt{(n-1)}\}}$$

Hence
$$\frac{n^k}{n \, . \, 2\sqrt{n}} < a_n < \frac{n^k}{\sqrt{(\frac{1}{2}n^2)} 2\sqrt{(\frac{1}{2}n)}}$$

that is
$$\tfrac{1}{2} n^{k-1\frac{1}{2}} < a_n < n^{k-1\frac{1}{2}}$$

Therefore by the comparison test, it follows from p. 67 that $\sum a_n$ converges if $k < \frac{1}{2}$ and diverges if $k \geqslant \frac{1}{2}$.

EXERCISE XIVc

A

For the values of a_n in Nos. 1-8, determine whether the series $\sum a_n$ are convergent or divergent.

1. $2^n/\sqrt{(4^n+1)}$ 2. $(n+1)/\sqrt{n^5}$ 3. $\sqrt{(1+n^2)} - n$

4. $n^b(n+1)^c$ for various values of b, c.

5. $(4 . 7 \ldots 3n+1)/(3 . 7 \ldots 4n-1)$ 6. $(n^b - n^c)^{-1}$, $0 < c < b$

7. $\{\sqrt{n} - \sqrt{(n-1)}\}/\sqrt{(n-1)}$ 8. $(\log n)/n$

9. If $a_n = 1/n$ when n is a square and otherwise $a_n = 1/n^2$, prove that $\sum a_n$ is convergent and that na_n does not tend to zero. Why does not Abel's Theorem (p. 339) apply to this series ?

10. If $\sum a_n^2$ is convergent, prove that $\sum\{|a_n|/n\}$ is also convergent.

B

For the values of a_n in Nos. 11-16. determine whether the series $\sum a_n$ are convergent or divergent.

11. $n(n-1)^q$, $q < 0$ 12. $(b^n - c^n)^{-1}$, $0 < c < b$

13. $n!/(3 . 5 . 7 \ldots 2n+1)$ 14. $2^n(n!)/(5 . 8 \ldots 3n+2)$

15. $\sqrt[3]{(2n^2-1)}/\sqrt[4]{(3n^3+2n+5)}$ 16. $\sqrt{(n^3+1)} - \sqrt{(n^3-1)}$

17. Prove that $\sum \log(1 + x^{2n})$ is convergent if $x^2 < 1$.

18. Prove that if $a_n > 0$, the series $\sum a_n$ and $\sum\{a_n/(1+a_n)\}$ are both convergent or both divergent.

C

19. If $a_n = \sqrt{n}\{\sqrt{(n+1)} - 2\sqrt{n} + \sqrt{(n-1)}\}$, determine whether the series $\sum a_n$ is convergent or divergent.

20. If $a_n = n \log\{(n+1)/(n-1)\} - 2$, determine whether the series $\sum a_n$ is convergent or divergent.

21. [*A Comparison Test*] If $\sum a_n$, $\sum b_n$ are two series of positive terms such that $a_n/a_{n+1} > b_n/b_{n+1}$ for all positive integral values of n and if $\sum b_n$ is convergent, prove that $\sum a_n$ is convergent.

22. [*Cauchy's Test*] If $a_n > 0$ and $\lim \sqrt[n]{a_n} = l$, prove that $\sum a_n$ is convergent or divergent according as $l < 1$ or $l > 1$. Show by examples that if $l = 1$, $\sum a_n$ may be either convergent or divergent.

23. Prove that the sum of the reciprocals of any number of positive integers that do not involve the digit 0 (when written in the usual way in the scale of 10) is less than $9 + 9^2/10 + 9^3/10^2 + \dots$, so that the series

$$1 + \tfrac{1}{2} + \dots + \tfrac{1}{9} + \tfrac{1}{11} + \dots + \tfrac{1}{19} + \tfrac{1}{21} + \dots$$

is convergent with sum to infinity less than 90 although $\sum 1/n$ is divergent.

24. If $\sum a_n$ is a divergent series of positive terms and

$$A_n = a_1 + a_2 + \dots + a_n,$$

prove that $\sum(a_{n+1}/A_n)$ is also divergent.

Deduce that $\sum \left\{\log\left(1 + \frac{1}{n}\right)\Big/\log n\right\}$ is divergent.

25. If (a_n) is monotone and diverges to $+\infty$, prove that $\sum\{(a_{n+1} - a_n)/a_n\}$ is divergent. Deduce that $\sum\{1/(n \log n)\}$ is divergent.

26. If (a_n) is monotone and diverges to $+\infty$ and if $p > 1$, prove that $\sum\{(a_{n+1} - a_n)/(a_n{}^{p-1} a_{n+1})\}$ is convergent. Deduce that

$$\sum\{n^{-1}(\log n)^{-p}\}$$

is convergent if $p > 1$.

27. If (a_n) is monotone and diverges to $+\infty$, and if

$$s_r = a_1 - a_2 + a_3 - \dots$$

to r terms, prove that

(i) s_{2n+1} tends to a limit Λ or to $+\infty$,

(ii) s_{2n} tends to a limit λ or to $-\infty$,

(iii) Λ and λ cannot both exist.

Hence prove that $a_1 - a_2 + a_3 - \dots$ oscillates infinitely.

Prove that $2^2 - 2 + 2^3 - 2^2 + 2^4 - 2^3 + \dots$ diverges to $+\infty$.

Infinite Integrals. An *infinite integral* $\int_a^\infty f(x)dx$ is defined to be the limit, if it exists, of $\int_a^t f(x)dx$ when $t \to \infty$. If the limit exists, then the limit of s_n, $\equiv \int_a^n f(x)dx$, when n tends to infinity *through positive integral values* must also exist. Hence the convergence of the sequence (s_n) is *necessary* for the existence of the infinite integral. But it is *not sufficient*. For example if $s_n \equiv \int_0^n \cos 2\pi x\, dx$, the sequence is convergent because $s_n = 0$ for all values of n, but $\int_0^\infty \cos 2\pi x\, dx$ does not exist.

If however it can be shown that $\int_a^t f(x)dx$ lies between s_n and s_{n+1} whenever t lies between n and $n+1$, then the convergence of (s_n) is a *sufficient* condition for the existence of the integral.

The Integral Test. *If $f(x)$ is a* positive *integrable function of x which is* monotone decreasing *for $x > m$ where m is a given positive integer, then the series $\sum f(n)$ converges if and only if $\int_m^\infty f(x)dx$ exists.*

If r is an integer greater than m and $r-1 < x < r$, then

$$f(r) < f(x) < f(r-1)$$

$$\therefore \int_{r-1}^r f(r)dx < \int_{r-1}^r f(x)dx < \int_{r-1}^r f(r-1)dx$$

$$\therefore f(r) < \int_{r-1}^r f(x)dx < f(r-1)$$

Putting $r = m+1, m+2, \dots, n$ and adding,

$$f(m+1) + f(m+2) + \dots + f(n) < \int_m^n f(x)dx < f(m) + f(m+1) + \dots + f(n-1).$$

Thus $\sum_m^n f(r) < f(m) + \int_m^n f(x)dx$, and if $\int_m^n f(x)dx \to l$ when $n \to \infty$, $\sum_m^n f(r)$ is a monotone increasing function of n which never exceeds $f(m) + l$ and therefore converges.

And conversely if $\sum_m^n f(r) \to s$ when $n \to \infty$, $\int_m^n f(x)dx$ is a monotone increasing function of n which never exceeds s and so converges.

If $f(x)$ is a positive integrable function of x which is monotone decreasing, $\int_m^\infty f(x)dx$ *may not* exist even when $f(x) \to 0$ when $x \to \infty$. For example if $f(x) = \frac{1}{x}$, $\int_1^\infty \frac{1}{x} dx$ does not exist. If, when $x \to \infty$, $f(x) \to k > 0$, then $\int_m^\infty f(x)dx$ *cannot* exist because

$$\int_m^n f(x)dx \geqslant (n-m)k.$$

But it follows from the inequalities obtained above that if

$$t_n \equiv f(m) + f(m+1) + \ldots + f(n-1) - \int_m^n f(x)dx,$$

the sequence (t_n) is convergent.

For since $f(n) > \int_n^{n+1} f(x)dx$, t_n is a monotone increasing function of n; and it has been proved that $t_n < f(m) - f(n) < f(m)$ $\therefore$ (t_n) converges to a limit $< f(m)$.

For an illustration of this property, see Example 11 (ii). See also Example 1, p. 330.

Example 11. If $a_n = \dfrac{1}{(n+1)\log(n+1)}$ and $A_n \equiv a_1 + a_2 + \ldots + a_n$ prove that (i) the sequence (A_n) is divergent,

and (ii) the sequence $(A_n - \log\log n)$ is convergent.

(i) $\displaystyle\int_2^n \frac{dx}{x \log x} = \Big[\log\log x\Big]_2^n = \log\log n - \log\log 2.$

Hence $\displaystyle\int_2^\infty \frac{dx}{x\log x}$ does not exist. Also $\dfrac{1}{x \log x}$ is a monotone decreasing positive function. Hence (A_n) is not convergent, and since it is monotone increasing, it is divergent.

(ii) $A_{n-2} - \displaystyle\int_2^n \frac{dx}{x\log x}$ tends to a limit when $n \to \infty$.

But $a_r \to 0$ when $r \to \infty$, $\therefore$ $(A_n - \log\log n)$ is convergent.

Example 12. Prove that $\sum_{2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.

$$\int_2^n \frac{dx}{x(\log x)^2} = \left[\frac{-1}{\log x}\right]_2^n = \frac{1}{\log 2} - \frac{1}{\log n}$$

$\therefore \int_2^{\infty} \frac{dx}{x(\log x)^2}$ exists. Hence since $\frac{1}{x\,(\log x)^2}$ is a monotone decreasing positive function, $\sum \frac{1}{n(\log n)^2}$ is convergent.

Cauchy's Condensation Test. *If* $f(n)$ *is a* **positive monotone decreasing** *function of* n, *the series* $\sum f(n)$ *is convergent or divergent according as the series* $\sum 2^n f(2^n)$ *is convergent or divergent.*

$$\text{Put} \quad s_n = \sum_1^n f(r) \qquad t_n = \sum_1^n 2^r f(2^r).$$

(i) Suppose (t_n) converges to the limit t.

$$\{f(2)+f(3)\} + \{f(4)+\ldots+f(7)\} + \ldots + \{f(2^{r-1}) + \ldots + f(2^r - 1)\}$$
$$< 2f(2) + 2^2 f(2^2) + \ldots + 2^{r-1} f(2^{r-1}) = t_{r-1}.$$

Thus if $2^p > n$, $s_n < t_{p-1} + f(1) < t + f(1)$.

Also (s_n) is monotone increasing, therefore it converges.

(ii) Suppose (t_n) is a divergent sequence.

$$f(2) + \{f(3)+f(4)\} + \{f(5)+\ldots f(8)\} + \ldots + \{f(2^{r-1}+1) + \ldots + f(2^r)\}$$
$$> f(2) + 2f(2^2) + 2^2 f(2^3) + \ldots + 2^{r-1} f(2^r) = \tfrac{1}{2} t_r.$$

Thus if $n > 2^p$, $s_n > \frac{1}{2} t_p$; therefore (s_n) diverges.

Cauchy's test may be applied to the series in Examples 11, 12.

(i) If $f(n) = 1/\{n \log n\}$, $n > 1$,

$$2^n f(2^n) = 2^n / (2^n \log 2^n) = 1/(n \log 2).$$

But $\sum(1/n)$ is divergent, therefore $\sum f(n)$ is divergent.

(ii) If $f(n) = 1/\{n(\log n)^2\}$,

$$2^n f(2^n) = 2^n / \{2^n (\log 2^n)^2\} = 1/\{n^2 (\log 2)^2\}.$$

But $\sum(1/n^2)$ is convergent, therefore $\sum f(n)$ is convergent.

Ratio Tests. It was pointed out on p. 70 that D'Alembert's test fails when $\lim (a_n/a_{n+1}) = 1$. We give here two further tests which suffice for most purposes and in particular are decisive when a_n/a_{n+1} can be expressed as the ratio of two polynomials in n.

Raabe's Test. *For a series of* positive *terms*

(i) *if* $a_n/a_{n+1} \geqslant 1 + \alpha/n$ *where* $\alpha > 1$, $\sum a_n$ *is convergent,*

(ii) *if* $a_n/a_{n+1} \leqslant 1 + 1/n$, $\sum a_n$ *is divergent,*

and the same results hold if a_n/a_{n+1} *satisfies the inequality only for* $n \geqslant m$ *where* m *is a constant. In* (i) α *must be a constant.*

(i) If $r \geqslant m$,
$$a_r/a_{r+1} \geqslant 1 + \alpha/r.$$
Thus
$$ra_r \geqslant ra_{r+1} + \alpha a_{r+1}$$
i.e.
$$ra_r - (r+1)a_{r+1} \geqslant (\alpha - 1)a_{r+1}.$$

Put $r = m, m+1, \ldots, n-1$ and add,

thus
$$ma_m - na_n \geqslant (\alpha - 1)(a_{m+1} + a_{m+2} + \ldots + a_n)$$
where $n \geqslant m+1$. Also $\alpha > 1$,

thus
$$a_{m+1} + a_{m+2} + \ldots + a_n \leqslant (ma_m - na_n)/(\alpha - 1)$$
$$< ma_m/(\alpha - 1).$$

Hence if $s_n = a_1 + a_2 + \ldots + a_n$, (s_n) is a monotone increasing sequence and $s_n \leqslant s_m + ma_m/(\alpha - 1)$ which is constant. Thus (s_n) is convergent.

(ii) If $r \geqslant m$,
$$ra_r \leqslant (r+1)a_{r+1},$$
$$\therefore \; na_n \geqslant (n-1)a_{n-1} \geqslant \ldots \geqslant ma_m \quad (n > m)$$

Thus $a_n \geqslant ma_m/n$. But $\sum 1/n$ is divergent and therefore by the comparison test $\sum a_n$ is divergent.

In practice it is often simpler to use these tests by evaluating a limit, but it may also happen that the tests can be used when the corresponding limits do not exist.

If $\lim_{n\to\infty} n(a_n/a_{n+1} - 1)$ exists and is l where $l > 1$, then m exists such that $n(a_n/a_{n+1} - 1) \geqslant \frac{1}{2}(l+1) > 1$ for $n > m$,
$$\therefore \; a_n/a_{n+1} \geqslant 1 + \alpha/n \text{ where } \alpha = \tfrac{1}{2}(l+1) > 1,$$
$$\therefore \; \sum a_n \text{ is convergent.}$$

If $\lim_{n\to\infty} n(a_n/a_{n+1} - 1)$ exists and is l where $l < 1$, then m exists such that $n(a_n/a_{n+1} - 1) < 1$ for $n > m$,
$$\therefore \; a_n/a_{n+1} < 1 + 1/n,$$
$$\therefore \; \sum a_n \text{ is divergent.}$$

Gauss' Test. *For a series of* positive *terms, if*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\beta_n}{n^2}$$

where $|\beta_n|$ *is less than a constant* C *for* $n > m$, *then* $\sum a_n$ *is divergent.*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{1}{n \log n} \cdot \frac{\beta_n \log n}{n}.$$

But $(\log n)/n \to 0$ when $n \to \infty$ by p. 332. Therefore m_1 exists such that $\left|\frac{\beta_n \log n}{n}\right| < \frac{C \log n}{n} < 1$ when $n > m_1 > m$.

Then $$\frac{a_n}{a_{n+1}} < 1 + \frac{1}{n} + \frac{1}{n \log n}$$

$\therefore\ na_n \log n < (n+1)a_{n+1} \log n + a_{n+1}$

$\therefore\ na_n \log n - (n+1)a_{n+1} \log (n+1) < a_{n+1}\{1 - (n+1) \log (1 + 1/n)\}$
< 0, by p. 108.

Thus $(na_n \log n)$ is monotone increasing for $n > m_1$

and so $$na_n \log n > m_1 a_{m_1} \log m_1 = b_1 > 0.$$

Thus $a_n > b_1/(n \log n)$. But by Example 11 $\sum 1/(n \log n)$ is divergent; therefore by the comparison test, $\sum a_n$ is also divergent.

For an alternative method of proof, see Exercise XIVd, No. 32.

The O-notation. It is often a convenience to use the symbol O which is defined as follows.

The statement $$f(n) = O\,g(n)$$

means that constants C, m exist such that

$$|f(n)| < Cg(n) \text{ whenever } n > m.$$

The tests of Raabe and Gauss may therefore be stated in the form

$$\textit{If } a_n > 0 \textit{ and } \frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)$$

then $\sum a_n$ *is convergent if* $\alpha > 1$ *and divergent if* $\alpha < 1$.

The ratio tests can be used in conjunction with the results on pp. 72, 75 to decide the convergence or divergence of series in which the terms are not all positive.

Example 13. If $a_n = \dfrac{1}{1+x^n}$, discuss the convergence of $\sum a_n$.

If $x>1$, $a_n < 1/x^n$; therefore $\sum a_n$ converges.

If $x=1$, $a_n = \frac{1}{2}$; therefore $\sum a_n$ diverges to $+\infty$.

If $|x|<1$, $a_n>0$ and $a_n \to 1$ when $n\to\infty$; therefore $\sum a_n$ diverges to $+\infty$, (p. 64).

If $x=-1$, a_n does not exist if n is odd.

If $x<-1$, put $x=-y$ so that $y>1$,

then $$|a_n| = 1/\{y^n + (-1)^n\} = b_n, \text{ say},$$

but $b_n>0$ and $\lim (b_n/b_{n+1}) = y > 1$, $\therefore$ $\sum b_n$ is convergent.

Thus $\sum a_n$ is absolutely convergent and therefore convergent, (p. 74).

Example 14. If $a_n = n^k x^n$, discuss the convergence of $\sum a_n$.

$$\text{If } x \neq 0, \quad \left|\frac{a_n}{a_{n+1}}\right| = \frac{n^k}{|x|(n+1)^k} \to \frac{1}{|x|} \text{ when } n\to\infty;$$

hence $\sum a_n$ is divergent for $x>1$ and absolutely convergent (and therefore convergent) for $|x|<1$, $x\neq 0$.

For $x=0$ the terms are all zero.

For $x=1$, $a_n = n^k$, $\therefore$ $\sum a_n$ is convergent if $k<-1$ and divergent if $k \geqslant -1$, (p. 67).

For $x<-1$, if $1<p<|x|$, m exists such that $|a_{r+1}|>p|a_r|$ whenever $r>m$; thus $|a_n|>p^{n-m}|a_m|$. Hence $|a_n|\to\infty$ when $n\to\infty$. But the terms are alternately positive and negative, and $|a_n|$ is monotone increasing; thus by No. 27 on p. 341 $\sum a_n$ oscillates infinitely.

For $x=-1$, $|a_n|\to\infty$ if $k>0$; therefore $\sum a_n$ oscillates infinitely for the same reasons as before.

For $x=-1$ and $k=0$, $\sum a_n$ oscillates finitely.

For $x=-1$, $-1 \leqslant k < 0$, $(|a_n|)$ is monotone decreasing and converges to zero. Therefore by p. 72, $\sum a_n$ is convergent, but it is not absolutely convergent because $\sum a_n$ is divergent for $x=1$, $k \geqslant -1$.

For $x=-1$, $k<-1$, $\sum a_n$ is absolutely convergent because it has been proved convergent for $x=1$, $k<-1$.

Example 15. Determine whether the series

$$\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2+\dots$$

is convergent or divergent.

$$a_n=\left\{\frac{1.3\dots(2n-1)}{2.4\dots(2n)}\right\}^2 \qquad \therefore \ \frac{a_n}{a_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^2$$

$$\therefore \ \frac{a_n}{a_{n+1}}=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}=\left(1+\frac{2}{n}\right)\left(1-\frac{1}{n}\right)+O\left(\frac{1}{n^2}\right)$$

$$=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right).$$

Hence by Gauss' test the series is divergent.

Infinite Products. If $A_n=a_1+a_2+\dots+a_n$ and (A_n) is a convergent sequence with limit A, then A is called the sum to infinity of the series $a_1+a_2+\dots$ and this series is called convergent. Similar definitions are given for products, with one qualification.

If $P_n=a_1a_2\dots a_n$ and (P_n) is a convergent sequence whose limit P is *not zero*, then P is called the value of the *infinite product* $a_1a_2\dots$ which is called *convergent.*

If $P_n\to+\infty$, the infinite product is called divergent and is said to diverge to $+\infty$. It is also called divergent and is said to *diverge to zero* when $P_n\to 0$ although the sequence (P_n) is convergent.

For example if $a_n=\dfrac{(n+1)^2}{n(n+2)}$, $a_1a_2\dots a_n=\dfrac{2(n+1)}{n+2}\to 2$, and so $a_1a_2\dots$ is convergent with value 2. But the product $\frac{2}{1}\ \frac{3}{2}\ \frac{4}{3}\dots$ for which $a_n=\dfrac{n+1}{n}$ and therefore $a_1a_2\dots a_n=n+1$ is divergent to $+\infty$. Also the product $\frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}\dots$ for which $a_1a_2\dots a_n=1/(n+1)$ is divergent to zero, and so is the product $0.1.2.3.4\dots$, although the product $1.2.3.4\dots$ is divergent to $+\infty$.

Just as the convergence of a series $\sum a_n$ implies that $\lim a_n=0$ although this is not sufficient for convergence, so the convergence of a product $a_1a_2\dots$ implies that $\lim a_n=1$. This would not hold if a product for which $p_n\to 0$ were called convergent, as illustrated above by $0.1.2.3.4\dots$

On account of the property $\lim a_n = 1$, it is convenient to change the notation and to consider products of the form

$$P_n \equiv (1+a_1)(1+a_2)\ \dots\ (1+a_n)$$

where $a_n \to 0$ when $n \to \infty$. A condition of convergence is given by the following theorem.

If $0 < a_r < 1$, *the infinite products* $\Pi(1+a_r)$ *and* $\Pi(1-a_r)$ *are both convergent or both divergent according as the series* $\sum a_r$ *is convergent or divergent.*

If $\sum a_r$ *is divergent,* $\Pi(1+a_r)$ *diverges to* $+\infty$ *and* $\Pi(1-a_r)$ *diverges to zero.*

Let $$P_n = \prod_1^n (1+a_r),\quad Q_n = \prod_1^n (1-a_r),\quad A_n = \sum_1^n a_r.$$

Then if (A_n) converges to the limit A, there is no loss of generality in supposing $0 < A < 1$ since this merely involves the omission of a finite number of terms in (A_n) and of a finite number of non-zero factors in (P_n), (Q_n).

Since $(1-a_1)(1-a_2) > 1-(a_1+a_2)$ and $1-a_3 > 0$,

$$(1-a_1)(1-a_2)(1-a_3) > \{1-(a_1+a_2)\}(1-a_3) > 1-(a_1+a_2+a_3),$$

and so on. Hence $Q_n > 1 - A_n > 1 - A > 0$.

Also $$(1-a_r)(1+a_r) = 1 - a_r^2 < 1 \text{ and } 1 - a_r > 0,$$

$$\therefore\ 1 + a_r < 1/(1-a_r).$$

Hence $$P_n < 1/Q_n < 1/(1-A).$$

But P_n is monotone increasing and Q_n is monotone decreasing.

Therefore (P_n) converges to a limit which is positive and $< 1/(1-A)$ and (Q_n) converges to a limit which is $\geqslant (1-A) > 0$.

Therefore the infinite products are both convergent.

If $\sum a_r$ is divergent, since it is evident that $P_n > 1 + A_n$, (P_n) diverges to $+\infty$. Hence $(1/P_n)$ converges to zero and since $0 < Q_n < 1/P_n$, $Q_n \to 0$ when $n \to \infty$, and therefore (Q_n) diverges to zero. Thus the infinite products are both divergent.

Example 16. Prove that $\sum \frac{4.7.10 \ldots 3n+1}{5.8.11 \ldots 3n+2}$ is divergent.

First Method. By p. 206, if s_n denotes the sum to n terms,

$$s_n = \frac{4.7.10 \ldots 3n+4}{2.5.\ 8 \ldots 3n+2} - 2.$$

But
$$\frac{3r+4}{3r+2} = 1 + \frac{2}{3r+2} > 1 + \frac{2}{3(r+1)}$$

and so $s_n > -2 + \left(1+\frac{2}{3}1\right)\left(1+\frac{2}{3}\frac{1}{2}\right)\left(1+\frac{2}{3}\frac{1}{3}\right) \ldots \left(1+\frac{2}{3}\frac{1}{n+1}\right).$

But $\sum(1/n)$ diverges; hence $\Pi\left(1+\frac{2}{3}\frac{1}{r}\right)$ diverges; therefore $s_n \to \infty$ when $n \to \infty$.

Second Method. If a_n denotes the n^{th} term of the series,

$$\frac{a_n}{a_{n+1}} = \frac{3n+5}{3n+4} = 1 + \frac{1}{3n+4} < 1 + \frac{1}{n}$$

and hence by Raabe's test, $\sum a_n$ is divergent.

Example 17. If p_r is the r^{th} prime number, that is the r^{th} term of the series 2, 3, 5, 7, 11, ... , prove that $\Pi(1-1/p_r)$ diverges to zero and $\sum(1/p_r)$ diverges to $+\infty$.

If $p_n = x$ and m is a positive integer such that $2^{m+1} > x$, any positive integer less than or equal to x can be expressed in prime factors in the form $\prod_{r=1}^{n} p_r{}^s$ where $p_r < x$ and $0 < s < m$, and therefore the product $\prod_{r=1}^{n} (1 + p_r{}^{-1} + p_r{}^{-2} + \ldots + p_r{}^{-m})$ contains each of the terms $1, 2^{-1}, 3^{-1}, 4^{-1}, \ldots, x^{-1}$.

Hence
$$\sum_{t=1}^{x} \frac{1}{t} < \prod_{r=1}^{n} (1 + p_r{}^{-1} + p_r{}^{-2} + \ldots + p_r{}^{-m})$$
$$= \prod_{1}^{n} \frac{1 - p_r{}^{-m-1}}{1 - p_r{}^{-1}} < \prod_{1}^{n} \frac{1}{1 - p_r{}^{-1}}.$$

Therefore when $n \to \infty$, $\prod_{1}^{n}(1 - p_r{}^{-1})^{-1} \to \infty$ and $\prod_{1}^{n}(1 - p_r{}^{-1}) \to 0$.

Hence $\Pi(1 - p_r{}^{-1})$ diverges to zero, and by the theorem on p. 349, $\sum p_r{}^{-1}$ is not convergent and therefore it diverges to $+\infty$.

Example 18. Discuss the convergence of the binomial series

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots \text{ for } x = +1,\ x = -1.$$

In the following discussion the trivial case $m=0$ is excluded since if $m=0$ every term except the first is zero and the series is therefore convergent.

If **x** $= -1$, the terms a_r, a_{r+1}, ... all have the same sign when $r > m+1$, and

$$\frac{a_r}{a_{r+1}} = \frac{r}{r-m-1} = 1 + \frac{m+1}{r-(m+1)} = 1 + \frac{m+1}{r} + O\left(\frac{1}{r^2}\right).$$

Hence by Raabe's test, *the series is convergent if* $m>0$ *and divergent if* $m<0$.

If **x** $= +1$, the terms a_r, a_{r+1}, ... have alternate signs when $r > m+1$ and $\left|\frac{a_r}{a_{r+1}}\right| = \frac{r}{r-m-1}$.

Hence if $m+1>0$, $|a_r|$ is a monotone decreasing sequence.

Also $$\left|\frac{a_{r+p+1}}{a_r}\right| = \frac{r-(m+1)}{r}\,\frac{r+1-(m+1)}{r+1}\cdots\frac{r+p-(m+1)}{r+p}$$

$$= \left(1 - \frac{m+1}{r}\right)\left(1 - \frac{m+1}{r+1}\right)\cdots\left(1 - \frac{m+1}{r+p}\right).$$

But this product diverges to zero if $m+1>0$ and diverges to $+\infty$ if $m+1<0$. Hence $(|a_n|)$ converges to zero if $m+1>0$, and diverges to $+\infty$ if $m+1<0$. Thus by p. 72 the series *converges if* $m>-1$. If $m<-1$, $|a_n|$ is monotone and diverges to $+\infty$; by No. 27, p. 341, the series *oscillates infinitely.*

If $m=-1$, the series is $1-1+1-1+\ldots$ It therefore *oscillates finitely if* $m=-1$.

It follows from what has been proved for $x=-1$, that when $x=+1$ the convergence is *absolute for* $m \geqslant 0$ *and conditional for* $-1<m<0$.

EXERCISE XIVd

A

For the values of a_n in Nos. 1-10, investigate the convergence of the series $\sum a_n$. Apply both the integral test and Cauchy's condensation test in Nos. 1, 2.

1. $\dfrac{1}{n(\log n)^3}$ 2. $\dfrac{1}{n \log n \log \log n}$ 3. $\dfrac{(\log n)^2}{n^2}$

4. $\dfrac{1}{n} x^{[\frac{1}{2}n]}$ where $[p]$ denotes the greatest integer not greater than p.

5. $\left(1 - \dfrac{1}{n}\right)^{n^2}$ 6. $\dfrac{(n+1)x^n}{n^2}$ 7. $c^{1/n} - 1$, $c > 0$.

8. $\dfrac{1.3.5 \ldots 2n-3}{2.4.6 \ldots 2n-2} \dfrac{x^{2n-1}}{2n-1}$ 9. $\dfrac{n!}{a(a+1) \ldots (a+n)}$, $a > 0$.

10. (i) $f(n)/\sqrt{n}$, (ii) $f(n)/(n+1)$, (iii) $f(n)/(2n+1)$, where

$$f(n) = (1.3 \ldots 2n-1)/(2.4 \ldots 2n).$$

11. If $b_r = 1 - 1/r^2$, verify that $\prod_2^n b_r$ is convergent with limit $\frac{1}{2}$.

12. Investigate the convergence of $\prod \dfrac{r^2 + ar + b}{r^2 + cr + d}$

B

For the values of a_n in Nos. 13-21, investigate the convergence of the series $\sum a_n$.

13. $1/\{n\sqrt{(\log n)}\}$ 14. $n!/n^n$ 15. $(\log n)^{-n}$

16. $a_n = x^n$ if n is odd, $a_n = y^n$ if n is even, $0 < x < y < 1$.

17. $(2^n - 2)x^{2n}/(2^n + 1)$ 18. nx^{2n^2} 19. $4^n(n!)^2/(2n+1)!$

20. $\{2.6 \ldots (4n-2)\}/\{3.7 \ldots (4n+3)\}$

21. $(1.4 \ldots 3n-2)^3/(2.5 \ldots 3n-1)^3$

22. Investigate the convergence of $\prod \left(\dfrac{x + x^{2r}}{1 + x^{2r}}\right)$

C

For the values of a_n in Nos. 23-25, investigate the convergence of the series $\sum a_n$.

23. (i) $(n \log n)^{-1}(\log \log n)^{-2}$, (ii) $(\log \log n)^3/\{n(\log n)^2\}$

24. $\dfrac{n^2-1}{(n^3+1)(\log n)^2}$ 25. $\left\{\dfrac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}\right\}^p$

26. Prove that (i) $\sum_1^\infty \dfrac{1}{n^2} < 2$, (ii) $\sum_1^\infty \dfrac{1}{n\sqrt{n}} < 3$

27. If a, b are positive, prove that

$$\sum_{r=1}^{n} \frac{1}{a+(r-1)b} - \frac{1}{b}\log(a+nb)$$

tends to a limit when $n \to \infty$.

28. Examine the convergence of $\Pi(1+a_r)$ where

(i) $a_r = (-1)^{r-1}/(r+1)$, (ii) $a_r = (-1)^{r-1}/\sqrt{(r+1)}$.

29. If for n integral and greater than 1,

$$a_{2n-1} = -1/n^p, \quad a_{2n} = 1/n^p + 1/n^{2p}$$

where $\frac{1}{2} > p > \frac{1}{3}$, prove that $\sum a_n$ and $\sum a_n^2$ are both divergent and that $\Pi(1+a_n)$ is convergent.

30. If $f(x) > 0$, $f'(x) > 0$, $f''(x) < 0$, prove that the series $\sum f'(n)$ and $\sum\{f'(n)/f(n)\}$ both converge or both diverge. If in addition $f(x) \to \infty$ when $x \to \infty$, prove that the series $\sum\{f'(x)/[f(x)]^\alpha\}$ is convergent for $\alpha > 1$.

31. If $a_n/a_{n+1} = 1 + b_n/n$ where $b_n \to b > 0$ when $n \to \infty$, prove that m exists so that $a_m/a_{n+1} > 1 + \frac{1}{2}b\sum_m^n(1/r)$ and deduce that $a_n \to 0$ when $n \to \infty$.

32. If $\sum a_n$ is a series of positive terms and $\sum(1/b_n)$ is a divergent series of positive terms, prove that $\sum a_n$ is convergent if

$$b_r a_r/a_{r+1} - b_{r+1} > k > 0$$

for all values of r and is divergent if $b_r a_r/a_{r+1} \leqslant b_{r+1}$.

Deduce Raabe's test by taking $b_n = n$, and establish the more general form of Gauss' test that $\sum a_n$ is divergent if

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + O\left(\frac{1}{n^{1+p}}\right)$$

where $p > 0$, by taking $b_n = n \log n$.

33. With the notation of Example 17, prove that

$$\log\left\{\prod_{r=1}^{n}\left(1-\frac{1}{p_r}\right)^{-1}\right\} - \sum_{r=1}^{n}\frac{1}{p_r} < \frac{1}{2}$$

and deduce that $\sum_{r=1}^{n}\dfrac{1}{p_r} > \log\log p_n - \dfrac{1}{2}$.

Absolute Convergence. The definition is given on p. 74, and the property, there proved, that an absolutely convergent series is convergent has already been used in this chapter. See Examples 13, 14, p. 347. The manipulation of an absolutely convergent series is essentially simpler than that of a conditionally convergent series on account of the following theorem.

Dirichlet's Theorem. *The sum to infinity of a convergent series of positive terms or of any other* **absolutely convergent** *series is unaffected by a change in the order of the terms.*

(i) First suppose that $a_1 + a_2 + a_3 + \dots$ is a convergent series of positive terms and denote its sum to infinity which is $\lim A_n$ by A. Consider a new series $b_1 + b_2 + b_3 + \dots$ formed by rearranging the terms of $\sum a_r$. Then if n is given, another integer m exists such that

$$B_n < A_m < A.$$

(B_n) is therefore a monotone increasing sequence whose terms do not exceed A. Hence it converges to a limit B and $B \leqslant A$.

Since $a_1 + a_2 + a_3 + \dots$ can be obtained by rearranging the terms of $b_1 + b_2 + b_3 + \dots$, it may be proved similarly that $A \leqslant B$. Hence $A = B$.

(ii) Now suppose that $a_1 + a_2 + a_3 + \dots$ is an absolutely convergent series with sum to infinity A and that the sum to infinity of $|a_1| + |a_2| + |a_3| + \dots$ is A'.

It follows from p. 64 that $\{a_1 + |a_1|\} + \{a_2 + |a_2|\} + \dots$ converges and has $A + A'$ for sum to infinity.

If $b_1 + b_2 + b_3 + \dots$ is the rearranged series, then

$$|b_1| + |b_2| + |b_3| + \dots$$

is a rearrangement of $|a_1| + |a_2| + |a_3| + \dots$ and therefore by (i) it converges to A'.

Also $\{b_1 + |b_1|\} + \{b_2 + |b_2|\} + \dots$ is a series with no negative terms and is a rearrangement of $\{a_1 + |a_1|\} + \{a_2 + |a_2|\} + \dots$. Therefore by (i) it converges to $A + A'$.

It now follows from p. 64 that $b_1 + b_2 + b_3 + \dots$ converges to $(A + A') - A'$, that is to A.

In contrast to the property of Dirichlet's theorem, if the terms of a conditionally convergent series are rearranged, the new series may have a different sum (see Example 19) or may fail to converge (see Exercise XIVg, No. 6).

It is not in fact difficult to prove that the terms may be rearranged so that the new series diverges, oscillates, or has a given sum to infinity. This is known as Riemann's Theorem; for a proof the reader is referred to Bromwich: *Infinite Series*, p. 68 (1st edition). In the following example the series has the same terms as the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ for log 2, but they are arranged in a different order, and it is shown that the sum to infinity is $\frac{1}{2}\log 2$.

Example 19. Find the sum to infinity of

$$1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{5}-\tfrac{1}{10}-\tfrac{1}{12}+\ldots$$

Denote the sum to r terms by s_r.

Then

$$s_{3n}=\{(1-\tfrac{1}{2})-\tfrac{1}{4}\}+\{(\tfrac{1}{3}-\tfrac{1}{6})-\tfrac{1}{8}\}+\ldots+\left\{\left(\frac{1}{2n-1}-\frac{1}{4n-2}\right)-\frac{1}{4n}\right\}$$

$$=\{\tfrac{1}{2}-\tfrac{1}{4}\}+\{\tfrac{1}{6}-\tfrac{1}{8}\}+\ldots+\left\{\frac{1}{4n-2}-\frac{1}{4n}\right\}$$

$$=\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\ldots \text{ to } 2n \text{ terms})$$

$$\therefore \lim_{n\to\infty} s_{3n}=\tfrac{1}{2}\log 2.$$

But it is necessary also to consider the limits of s_{3n+1} and s_{3n+2}.

Since

$$s_{3n+1}-s_{3n}=\frac{1}{2n+1}$$

$$\therefore \lim_{n\to\infty} s_{3n+1}=\lim_{n\to\infty} s_{3n}=\tfrac{1}{2}\log 2$$

Similarly

$$s_{3n+2}-s_{3n}=\frac{1}{4n+2}$$

$$\therefore \lim_{n\to\infty} s_{3n+2}=\tfrac{1}{2}\log 2.$$

Hence

$$\lim_{r\to\infty} s_r=\tfrac{1}{2}\log 2.$$

Cauchy's Multiplication Theorem. *If* $a_1 + a_2 + a_3 + \ldots$ *and* $b_1 + b_2 + b_3 + \ldots$ *are convergent series of positive terms or* **absolutely convergent** *series, with sums to infinity A and B, then the series*

$$a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \ldots \quad \ldots\ldots\ldots\text{(i)}$$

is also convergent and its sum to infinity is AB. Also the convergence is absolute.

Let A', B' be the sums to infinity of

$$|a_1| + |a_2| + \ldots, \quad |b_1| + |b_2| + \ldots$$

Consider also the series whose sum to n terms is

$$(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n).$$

This is convergent and has AB for sum to infinity. It may be written

$$(a_1 b_1) + (a_1 b_2 + a_2 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_3 + a_3 b_3 + a_3 b_2 + a_3 b_1) + \ldots\text{(ii)}$$

where the n^{th} bracket contains all the products involving a_n or b_n but not involving any suffix greater than n.

The series obtained by omitting brackets, namely

$$a_1 b_1 + a_1 b_2 + a_2 b_2 + a_2 b_1 + a_1 b_3 + a_2 b_3 + a_3 b_3 + a_3 b_2 + a_3 b_1 + \ldots\text{(iii)}$$

is absolutely convergent, because

$$|a_1 b_1| + |a_1 b_2| + |a_2 b_2| + |a_2 b_1| + |a_1 b_3| + \ldots \text{ to } k \text{ terms}$$
$$\leqslant (|a_1| + |a_2| + \ldots + |a_k|)(|b_1| + |b_2| + \ldots + |b_k|) \leqslant A'B'.$$

Hence all series formed by rearranging the terms of (iii) have the same sum to infinity. One of these is the series (ii) whose sum is AB, and another is the series (i). Hence the series (i) converges to AB.

The convergence of (i) is absolute because

$$|a_1 b_1| + |a_1 b_2 + a_2 b_1| + |a_1 b_3 + a_2 b_2 + a_3 b_1| + \ldots$$

is a series of positive terms whose sum to n terms does not exceed $A'B'$.

Double Series. An arrangement

$$\begin{array}{l} a_{11}+a_{12}+a_{13}+\ldots \\ +a_{21}+a_{22}+a_{23}+\ldots \\ +a_{31}+a_{32}+a_{33}+\ldots \\ +\ldots\ldots\ldots \\ \quad\vdots\quad\vdots\quad\vdots \end{array}$$

in which the term of the p^{th} row in the q^{th} column is a_{pq}, is called a *double series*. No meaning has yet been assigned in this book to ‘ the sum to infinity of a double series ’.

If for each value of r, $a_{r1}+a_{r2}+a_{r3}+\ldots$ is convergent with sum to infinity A_r, and if $A_1+A_2+A_3+\ldots$ is convergent with sum to infinity A, then A is called the *sum by rows* of the double series.

If for each value of s, $a_{1s}+a_{2s}+a_{3s}+\ldots$ is convergent with sum to infinity B_s, and if $B_1+B_2+B_3+\ldots$ is convergent with sum to infinity B, then B is called the *sum by columns* of the double series.

Further if $a_{11}+(a_{12}+a_{21})+(a_{13}+a_{22}+a_{31})+\ldots$ is convergent, its sum to infinity is called the *sum by diagonals* of the double series ;

and if $a_{11}+(a_{12}+a_{22}+a_{21})+(a_{13}+a_{23}+a_{33}+a_{32}+a_{31})+\ldots$

is convergent, its sum to infinity is called the *sum by squares*, and other modes of summation can be defined.

There is no reason to suppose that the various sums of a double series which may happen to exist are necessarily equal. And in the example

$$\begin{array}{l} 0+1+0+0+0+0+0+\ldots \\ -1+0+1+0+0+0+0+\ldots \\ +0-1+0+1+0+0+0+\ldots \\ +0+0-1+0+1+0+0+\ldots \\ +0+0+0-1+0+1+0+\ldots \\ \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots \end{array}$$

the sum by rows is 1, the sum by columns is -1, and the sums by diagonals and by squares are both zero.

The general discussion of double series is beyond the scope of this book. One result has however just been proved, since Cauchy's Multiplication Theorem may be enunciated in the following form.

In the double series defined by $a_{pq}=a_p b_q$, if $a_1+a_2+a_3+\dots$ and $b_1+b_2+b_3+\dots$ are absolutely convergent and have A, B as sums to infinity (so that the sums by rows and by columns are each AB), then the sum by diagonals is AB.

The proof also shows that the sum by squares is AB.

It is possible to prove by Dirichlet's theorem that if all the terms of a double series are positive and if one of the sums exists, then the others exist and all the sums are equal. In this case *the double series is called convergent.*

EXERCISE XIVe

A

Determine the conditions for convergence, conditional or absolute, divergence, and oscillation of the series in Nos. 1-4.

1. $\Sigma \dfrac{1}{1+nx}$ 2. $\Sigma \dfrac{x^n}{1+x^n}$ 3. $\Sigma \dfrac{n^2-1}{n^2+1} x^n$

4. $\dfrac{1}{2}-\dfrac{1}{2.4}x+\dfrac{1.3}{2.4.6}x^2-\dfrac{1.3.5}{2.4.6.8}x^3+\dots$

Find $\Sigma\Sigma a_{pq}$ by rows, by columns, by diagonals, and by squares, in Nos. 5-7.

5. $a_{pq}=x^{p+q-1}$, $0<x<1$.

6. $a_{1r}=a_{r1}=1$; $a_{2r}=a_{r2}=-1$ if $r>1$; otherwise $a_{pq}=0$.

7. $a_{rr}=2$; $a_{r,r+2}=a_{r+2,r}=-1$; otherwise $a_{pq}=0$.

8. Write down the square of $1+\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\dots$ as a double series. Show that its sum by diagonals is $\Sigma r 2^{1-r}$, and evaluate this sum. Find also the sum by rows.

9. If the terms of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ are rearranged in the order $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\dots$, prove that the sum becomes $\frac{3}{2}\log 2$.

B

Discuss as in Nos. 1-4 the convergence of the series in Nos. 10-12.

10. $\sum x^n/(1+nx^{2n})$

11. $\sum 1/\{1+(n+1)x^{n+1}\}$

12. $\frac{1}{2}+\frac{1.3}{2.4}x+\frac{1.3.5}{2.4.6}x^2+\frac{1.3.5.7}{2.4.6.8}x^3+\dots$

Find $\sum\sum a_{pq}$ by rows, by columns, by diagonals, and by squares in Nos. 13, 14.

13. $a_{pq}=x^p y^q$, $0<x<y<1$.

14. $a_{rr}=2$, $a_{r,r+1}=a_{r+1,r}=-1$, otherwise $a_{pq}=0$.

C

15. If $\sum a_n$, $\sum b_n$ are convergent series of positive terms, prove that $\sum a_n b_n$ is convergent.

16. If a double series of positive terms has a sum by squares prove that it also has a sum by diagonals and that these sums are equal. Also prove the converse.

17. Discuss as in Nos. 1-4 the convergence of

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\dots$$

18. Find $\sum\sum a_{pq}$ by rows, by columns, and by squares if

$$a_{pq}=p^q(p+1)^{-q-1}-(p+1)^q(p+2)^{-q-1}$$

19. If the terms of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ are rearranged in the order $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\dots$, prove that the sum becomes $\frac{1}{2}\log 12$.

20. Prove that the series

$$1-\tfrac{1}{2}(1+\tfrac{1}{2})+\tfrac{1}{3}(1+\tfrac{1}{2}+\tfrac{1}{3})-\tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+\dots$$

is convergent.

Infinite Continued Fractions. The convergents of the continued fraction $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 + \ldots}$ are given (p. 244) by

$$p_r = a_r p_{r-1} + p_{r-2} \qquad q_r = a_r q_{r-1} + q_{r-2}.$$

It is assumed that $a_1, a_2, a_3, \ldots$ are positive integers except that a_1 may be zero $\therefore\ p_r > p_{r-1} + 1,\ r > 2.$

Hence the sequence (p_r) is monotone increasing in the strict sense and diverges to $+\infty$. Similarly for q_r.

Also from p. 245, $$\frac{p_r}{q_r} - \frac{p_{r-1}}{q_{r-1}} = \frac{(-1)^r}{q_r q_{r-1}}$$

Putting $r = 2, 3, \ldots, n$ and adding,

$$\frac{p_n}{q_n} - \frac{p_1}{q_1} = \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \ldots + (-1)^n \frac{1}{q_{n-1} q_n}$$

In this series, the terms have alternate signs and decrease in absolute value; also the n^{th} term tends to zero when $n \to \infty$. Hence by p. 72, the series is convergent. Thus p_n/q_n tends to a limit F. This limit is called the value of the infinite continued fraction. It lies between the values of p_n/q_n for any two consecutive values of n. This gives a limit to the error when a convergent is used as an approximation to the value of the fraction.

Example 20. Express $\sqrt{2}$ as a continued fraction and deduce its value to 3 places of decimals.

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + 1/x \text{ where } x = 1/(\sqrt{2} - 1) = 1 + \sqrt{2}.$$

Hence $$\sqrt{2} = 1 + \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 + \ldots}$$

The convergents are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$.

It follows that $\frac{41}{29} = 1{\cdot}4137 \ldots < \sqrt{2} < \frac{99}{70} = 1{\cdot}41428 \ldots$ and so $\sqrt{2} = 1{\cdot}414$ to 3 places of decimals.

Example 21. Evaluate $\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\ldots$

By p. 360, p_r/q_r tends to a limit, say F. Also

$$p_{2n+2}/q_{2n+2}=\frac{1}{2+}\frac{1}{4+}\frac{p_{2n}}{q_{2n}}$$

$$\therefore\ F=\frac{1}{2+}\frac{1}{4+}\frac{F}{1}=\frac{4+F}{9+2F}$$

whence $$(F+2)^2=6.$$

But F is positive, $\therefore\ F=\sqrt{6}-2$.

See Example 14, p. 243.

Example 22. Evaluate $2+\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\ldots$

If $x=\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\ldots$ as in Example 21

$$x=\frac{1}{1+}\frac{1}{2+}\frac{x}{1}=\frac{2+x}{3+x}$$

Thus $x^2+2x=2$; but $x>0$, $\therefore\ x=\sqrt{3}-1$.

Hence the value of the given fraction $=2+\dfrac{1}{3+x}=4-\sqrt{3}$.

Example 23. Evaluate $1+\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\frac{4}{4+}\ldots$

Since $p_{r+1}=rp_r+rp_{r-1}$ and $q_{r+1}=rq_r+rq_{r-1}$, p_{r+1} and q_{r+1} are values of u_{r+1} which satisfy the difference equation

$$u_{r+1}-ru_r-ru_{r-1}=0,\quad (r>1).$$

From Example 9, pp. 232, 233, the solution is

$$\frac{u_n}{n!}=u_1+(u_2-2u_1)\sum_2^n\frac{(-1)^r}{r!}$$

But $p_1=1$, $q_1=1$, $p_2=2$, $q_2=1$;

hence $\dfrac{p_n}{n!}=p_1=1$ and $\dfrac{q_n}{n!}=1-\left\{\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots+(-1)^n\dfrac{1}{n!}\right\}$

$$\therefore\ \text{when } n\to\infty,\ \frac{q_n}{n!}\to 1-\frac{1}{e}\ \text{and}\ \frac{p_n}{q_n}\to\frac{e}{e-1}$$

$\therefore$ the value of the infinite continued fraction is $e/(e-1)$.

Example 24. If a, b, c are positive, find the value of x/y when

$$x = a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots$$

and

$$y = c + \frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{c+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots$$

From $x = a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{x}$, it follows that x is the positive root of the equation $x^2(bc+1) - x(abc+a+c-b) - (ab+1) = 0$. Hence by exchanging a and c, y must be the positive root of the equation $y^2(ba+1) - y(abc+c+a-b) - (cb+1) = 0$. Therefore $-1/y$ satisfies the same equation as x, and is the negative root.

Thus $\frac{x}{y} = -$ product of the roots $= \frac{ab+1}{bc+1}$

EXERCISE XIVf

A

Express Nos. 1-4 as infinite continued fractions.

1. $\sqrt{3}$ 2. $\sqrt{11}$ 3. $\sqrt{9\frac{3}{5}}$

4. $\sqrt{(n^2+1)}$ where n is a positive integer.

Evaluate the infinite continued fractions in Nos. 5-8.

5. $2 + \frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\cdots$ 6. $2 + \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+}\cdots$

7. $1 + \frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots$ 8. $a + \frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\cdots$

B

Express Nos. 9-12 as infinite continued fractions.

9. $\sqrt{26}$ 10. $3 - \sqrt{5}$

11. $\sqrt{(n^2+2)}$, n being a positive integer.

12. Each root of $6x^2 + 2x - 1 = 0$.

13. Express $\sqrt{15}$ as a continued fraction. Evaluate the 5th convergent and show that it gives the value of $\sqrt{15}$ correct to 3 places of decimals.

14. Evaluate $2 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$

C

15. If $a_1, a_2, \dots, b_2, b_3, \dots$ are positive integers, prove that the odd convergents of $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots$ form a monotone increasing sequence and the even convergents a monotone decreasing sequence. Deduce that (p_n/q_n) converges or oscillates.

16. If $x=a+\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\dots$

and $y=b+\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\dots$

prove that $x/y=a/b$.

17. If $x=a+\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\dots$

$y=b+\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\dots$

$z=c+\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c}+\dots$

prove that $xyz=t+\frac{1}{t+}\frac{1}{t+}\dots$ where $t=a+b+c+abc$.

Prove that the infinite continued fractions in Nos. 18-20 are convergent and find their values.

18. $\frac{1}{2-}\frac{2}{3-}\frac{3}{4-}\frac{4}{5-}\dots$

19. $\frac{1}{3-}\frac{2}{4-}\frac{3}{5-}\frac{4}{6-}\dots$

20. $\frac{1}{2+}\frac{1.2}{2}+\frac{2.3}{2}+\frac{3.4}{2}+\dots$

21. If $\frac{p_r}{q_r}$ is the r^{th} convergent of $a_1+\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots$ where $a_1, a_2, \dots, b_2, b_3 \dots$ are positive integers, prove that

(i) $\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=(-1)^n\frac{b_2b_3\dots b_n}{q_nq_{n-1}}$

(ii) $q_n>(a_na_{n-1}+b_n)q_{n-2}$

(iii) $\frac{q_nq_{n-1}}{b_2b_3\dots b_n}>\frac{a_2}{b_2}\prod_2^{n-1}\left(1+\frac{a_ra_{r+1}}{b_{r+1}}\right)$

Deduce that p_r/q_r tends to a limit when $r\to\infty$ if $\sum(a_{r-1}a_r/b_r)$ is divergent.

22. If P/Q, P'/Q' are the n^{th} and $(n-1)^{\text{th}}$ convergents of $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\dots\frac{1}{a_n}$ and F is the value of the infinite continued fraction $a_1 + \frac{1}{a_2+}\dots\frac{1}{a_n+}\frac{1}{a_1+}\dots\frac{1}{a_n+}\frac{1}{a_1+}\dots$, where $a_1, a_2, \dots$ are positive integers, prove that

(i) F is the positive root of $f(x) \equiv x^2Q + x(Q'-P) - P' = 0$;

(ii) P/P', Q/Q' are the n^{th} and $(n-1)^{\text{th}}$ convergents of

$$a_n + \frac{1}{a_{n-1}+}\dots\frac{1}{a_1};$$

(iii) if F' is the value of the infinite continued fraction

$$a_n + \frac{1}{a_{n-1}+}\dots\frac{1}{a_1+}\frac{1}{a_n+}\dots$$

$-1/F'$ is the negative root of $f(x)=0$.

23. The infinite series $c_1 + c_2 + c_3 + \dots$ and the infinite continued fraction $\frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\dots$ are said to be equivalent if for every value of n, $C_n = p_n/q_n$ where $C_n = c_1 + c_2 + \dots + c_n$ and

$$\frac{p_n}{q_n} = \frac{b_1}{a_1+}\frac{b_2}{a_2+}\dots\frac{b_n}{a_n}.$$

(i) By finding the value of $p_n/q_n - p_{n-1}/q_{n-1}$ show that the continued fraction is equivalent to the series

$$\frac{b_1}{q_1} - \frac{b_1b_2}{q_1q_2} + \frac{b_1b_2b_3}{q_2q_3} - \dots + (-1)^{n-1}\frac{b_1b_2\dots b_n}{q_{n-1}q_n} + \dots$$

(ii) Construct a continued fraction equivalent to the infinite series $c_1 + c_2 + c_3 + \dots$ and such that $q_n = 1$ for all values of n, by solving the equations $C_n = a_nC_{n-1} + b_nC_{n-2}$, $1 = a_n + b_n$ for a_n, b_n. Hence prove that the series $c_1 + c_2 + c_3 + \dots$ is equivalent to

$$\frac{c_1}{1-}\frac{c_2/c_1}{1+c_2/c_1-}\frac{c_3/c_2}{1+c_3/c_2-}\dots\frac{c_n/c_{n-1}}{1+c_n/c_{n-1}-}\dots$$

(iii) Use the result in (ii) to find the series equivalent to

$$\frac{\alpha_1}{1-}\frac{\alpha_2}{1+\alpha_2-}\dots\frac{\alpha_n}{1+\alpha_n-}\dots$$

and to deduce from $\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ Brouncker's expression for $\frac{1}{4}\pi$ as a continued fraction.

MISCELLANEOUS EXAMPLES

EXERCISE XIVg

A

Find the conditions for the series in Nos. 1-3 to be convergent.

1. $\sum(\log n)^p n^{-q}$

2. $\sum n^k\{\sqrt{n}-3\sqrt{(n+1)}+3\sqrt{(n+2)}-\sqrt{(n+3)}\}$

3. $\dfrac{ab}{cd}+\dfrac{a(a+1)b(b+1)}{c(c+1)d(d+1)}+\dfrac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)d(d+1)(d+2)}+\ldots$

where a, b, c, d are positive.

4. If $u_n=\dfrac{n-1}{n}x^n-\dfrac{n}{n+1}x^{n+1}$, find when $\sum u_n$ is convergent and give the sum to infinity.

5. Discuss the convergence of

$$\text{(i)}\ 2-\tfrac{5}{4}+\tfrac{8}{7}-\tfrac{11}{10}+\tfrac{14}{13}-\tfrac{17}{16}+\ldots$$

$$\text{(ii)}\ (2-\tfrac{5}{4})+(\tfrac{8}{7}-\tfrac{11}{10})+(\tfrac{14}{13}-\tfrac{17}{16})+\ldots$$

6. Prove that $\sum\left\{\dfrac{1}{\sqrt{(4n-3)}}+\dfrac{1}{\sqrt{(4n-1)}}-\dfrac{1}{\sqrt{(2n)}}\right\}$ is divergent. Deduce that the series

$$1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{9}}+\ldots$$

is divergent, although $1-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{4}}+\dfrac{1}{\sqrt{5}}-\ldots$ is convergent.

B

Prove that the series in Nos. 7-12 are convergent.

7. $\sum\left(\dfrac{1}{n}+\dfrac{3}{n+2}-\dfrac{4}{n+3}\right)$

8. $\sum\dfrac{n}{n^a+b}$, $a>2$.

9. $\sum\{\log(1+1/n)\}^n$

10. $\sum x^n(x^{2n}-1)^{-1}$, $x^2\neq 1$.

11. $\sum\{-1+n\log(2n+1)-n\log(2n-1)\}$

12. $\sum(x^n\sin n\alpha)$, $|x|<1$.

13. Discuss the series in No. 12 when $x^2=1$ and $\alpha\neq k\pi$.

C

14. Find the condition that $\sum (n \log n \log\log n)^{-1}(\log\log\log n)^{-p}$ is convergent.

15. Sum the series $\dfrac{x^2}{x^2-1} - \dfrac{x^2}{x^4-1} - \dfrac{x^4}{x^8-1} - \ldots - \dfrac{x^k}{x^{2k}-1}$ where $k = 2^{n-1}$ and $x^2 \neq 1$. Show that the infinite series is convergent and find its sum to infinity.

16. If $|x| > 1$ and $k = 2^{n-1}$ show that $\sum\limits_n k(x^k+1)^{-1}$ is convergent and find its sum to infinity.

17. Prove that $\sum \exp\{-(\log n)^\alpha\}$ converges if and only if $\alpha > 1$.

18. Evaluate $\dfrac{1^2}{3} - \dfrac{2^2}{5} - \dfrac{3^2}{7} - \dfrac{4^2}{9} - \ldots$

19. (i) If $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots$ and $a_n \to 0$ and $nb_n = a_1 + a_2 + \ldots + a_n$, prove that $b_n \geqslant b_{n+1}$ and that $b_n \to 0$.

(ii) Establish the convergence of the series

$$\frac{1}{\log 2} - \frac{1}{2}\left(\frac{1}{\log 2} + \frac{1}{\log 3}\right) + \frac{1}{3}\left(\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4}\right) - \ldots$$

20. If $\lim\limits_{n \to \infty} \dfrac{f(n+1)}{f(n)} = l$ and $0 < l < 1$, prove that m exists such that $f(n) < \{\tfrac{1}{2}(l+1)\}^{n-m} f(m)$ whenever $n \geqslant m$, and that $f(n) \to 0$ when $n \to \infty$.

Extend the result to $-1 < l < 0$ by proving that $|f(n)| \to 0$ when $n \to \infty$.

21. Show that if $0 < x < 1$, the series $1 + x + x^2 + \ldots$ is equivalent to the continued fraction $\dfrac{1}{1-}\,\dfrac{x}{1+x-}\,\dfrac{x}{1+x-}\,\dfrac{x}{1+x}\ldots$ and deduce that $\dfrac{x}{1+x-}\,\dfrac{x}{1+x-}\,\dfrac{x}{1+x-}\ldots$ is equal to x if $0 < x < 1$ and to 1 if $x \geqslant 1$.

22. Prove that prime numbers exist as large as we please containing the digit 0 when written in the usual way in the scale of ten.

23. Prove that if m is not a positive integer, the binomial series $1 + mx + \frac{1}{2}m(m-1)x^2 + \ldots$ diverges for $x < -1$ and oscillates infinitely for $x > +1$.

ANSWERS

Page 197 EXERCISE Xa

1. 60, 10.
2. $\sum a^6 + 6\sum a^5b + 15\sum a^4b^2 + 30\sum a^4bc + 20\sum a^3b^3 + 60\sum a^3b^2c + 90a^2b^2c^2$.
3. 360. 4. n^2 5. $3^n/n!$
6. $(n2^{n+1}+1)/\{(n+1)(n+2)\}$ 13. 120 14. 2142
15. 37380 16. -1840 18. $n(2n+1)$
20. $2 \div (n+1)(n+2)(n+3)$
23. $\{1-(1-x)^{n+1}(1+nx+x)\} \div \{(n+1)(n+2)x^2\}$
30. $\binom{n}{\frac{1}{2}r}$ or zero 31. $\dfrac{(n-1)!\,(n+1)!}{(2n+1)!}$ 32. $[2p]_n \div [p]_n$
33. (i) $nx(1+x)^{n-3}\{1+(3n-1)x+n^2x^2\}$.
(ii) $n(n-1)x^2(1+x)^{n-4}\{6+6nx+n(n+1)x^2\}$.

Page 207 EXERCISE Xb

1. $\frac{1}{4}\{1-1/(4n+1)\}$; $\frac{1}{4}$ 2. $\frac{1}{6}[\frac{1}{88}-1/\{(3n+8)(3n+11)\}]$; $\frac{1}{528}$
3. $n(3n^2+3n-2)$ 4. $\frac{1}{4}n(125n^3+450n^2+295n-198)$
5. $n(n+1)^2$ 6. $\frac{1}{4}n(n+1)(n+4)(n+5)$
7. $\frac{1}{2}n(n+1)^2(n+2)$ 8. $\frac{3}{4}+\dfrac{1}{2(n+1)}-\dfrac{5}{2(n+2)}$; $\frac{3}{4}$
9. $\frac{3}{4}-\dfrac{1}{2(n+1)}-\dfrac{1}{2(n+2)}$; $\frac{3}{4}$
10. $\frac{17}{36}-\dfrac{1}{6(n+1)}-\dfrac{1}{6(n+2)}-\dfrac{2}{3(n+3)}$; $\frac{17}{36}$
11. $1-\dfrac{5\,.\,7\,.\,9 \ldots 2n+3}{8\,.\,10\,.\,12 \ldots 2n+6}$
12. $\frac{1}{2}-\dfrac{3\,.\,5\,.\,7 \ldots 2n+1}{2\,.\,4\,.\,6 \ldots 2n+2}$ 13. $\frac{1}{6}-1/(9n+6)$; $\frac{1}{6}$

14. $\frac{1}{4}[\frac{1}{3} - 1/\{(2n+1)(2n+3)\}]$; $\frac{1}{12}$. 15. $\frac{1}{3}n(4n^2+30n+71)$.

16. $n(16n^3+48n^2+14n-33)$. 17. $\frac{1}{2}n(6n^2+9n+1)$.

18. $\frac{1}{2}n(8n^2-5n-1)$ 19. $\frac{1}{8} - \frac{1}{16(2n+1)} - \frac{3}{16(2n+3)}$; $\frac{1}{8}$

20. $\frac{1}{3} - \frac{1}{4(2n+1)} - \frac{1}{4(2n+3)}$; $\frac{1}{3}$

21. $\frac{5}{6} - \frac{4}{n+3} + \frac{1}{n+2}$; $\frac{5}{6}$

22. $\frac{7}{36} - \frac{1}{3(n+1)} + \frac{1}{6(n+2)} + \frac{1}{6(n+3)}$; $\frac{7}{36}$

23. $\frac{5 \cdot 8 \ldots 3n+2}{3 \cdot 6 \ldots 3n} - 1$ 24. $\frac{3}{4}\frac{8 \cdot 13 \ldots 5n+8}{4 \cdot 9 \ldots 5n+4} - \frac{3}{2}$

25. $\frac{1}{18} - 1/\{3(n+1)(n+2)(n+3)\}$

26. $\frac{1}{2d}\left\{\frac{1}{a(a+d)} - \frac{1}{(a+nd)(a+nd+d)}\right\}$

27. $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$.

28. $an(a+nd) + \frac{1}{3}n(n^2-1)d^2$.

29. $\frac{1}{6}n(n+1)(6n^2+4n-1)$.

30. $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$

31. $\frac{1}{12}n(n+1)(3n^2+31n+74)$. 32. $\frac{1}{10}n(n^2-1)(n+2)(2n+1)$.

33. $\frac{1}{20}n(n+1)(n+2)(n+3)(4n+1)$.

34. $n + \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$

35. $\frac{121}{7200} - \frac{1}{40(n+1)} - \frac{1}{40(n+2)} + \frac{7}{120(n+3)} + \frac{7}{120(n+4)}$
$- \frac{1}{15(n+5)}$

36. $\frac{19}{288} - \frac{1}{24(n+1)} - \frac{1}{24(n+2)} - \frac{7}{24(n+3)} + \frac{3}{8(n+4)}$

37. $\frac{1}{2-p}\left\{\frac{(n+1)!}{p(p+1)\ldots(p+n-1)} - 1\right\}$

38. $\frac{1}{p+1}\left\{1 - \frac{(-1)^n \cdot (n+1)!}{(p-1)(p-2)\ldots(p-n)}\right\}$

39. $\frac{2p}{2p-1}\left\{1 - (-1)^n \frac{(2p-2)(2p-4)\ldots(2p-2n)}{1 \cdot 3 \cdot 5 \ldots (2n-1)}\right\}$

40. $a_1 a_2 \dots a_n - 1$; $\dfrac{a}{a-b}\left\{\dfrac{(a+d_1)(a+d_2)\dots(a+d_n)}{(b+d_1)(b+d_2)\dots(b+d_n)} - 1\right\}$

41. (i) $\dfrac{a}{a-b+d}\left\{\dfrac{(a+d)(a+2d)\dots(a+nd)}{b(b+d)\dots(b+nd-d)} - 1\right\}$

(ii) $\dfrac{a}{a+b-d}\left\{1-(-1)^n\dfrac{(a-d)(a-2d)\dots(a-nd)}{b(b+d)\dots(b+nd-d)}\right\}$

Page 211 EXERCISE Xc

1. $3(r+1)(r+2)$ 2. $r!\,r$ 3. $-3 \div \{r(r+1)(r+2)(r+3)\}$
4. $2\sin\theta\cos 2r\theta$ 5. $6(r+1)$ 6. $a(x-1)^2 x^{r-1}$
7. $\frac{1}{2}n(n-1)+c$; $\frac{1}{2}n(n+1)$
8. $\frac{1}{3}n(n^2-1)+c$; $\frac{1}{3}n(n+1)(n+2)$
9. $c+(x-x^n)/(1-x)$; $(x-x^{n+1})/(1-x)$
10. $c-1/(4n+2)$; $\frac{1}{6}-1/(4n+6)$
11. $\frac{1}{2}(3n+2)u_n+c$; $\frac{1}{2}(3n+4)u_n-2$ 12. $(n+1)!\,n$
13. $2^{n+1}/(n+2)-1$ 14. $2r+1$ 15. $-1\div\{r!\,(r+2)\}$
16. $6(r+2)$ 17. $\log\{r(r+2)/(r+1)^2\}$
18. $\frac{1}{12}(4n-7)(4n-3)(4n+1)+c$; $\frac{1}{3}n(16n^2+12n-13)$
19. $c-n^{-1}$; $n/(n+1)$ 20. $c+\log n$; $\log(n+1)$
21. $1-\{2^{n+1}/(n+2)!\}$ 22. $2-2^{n+1}\div\{3^n(n+1)\}$
23. $c+2\cos\frac{1}{2}\theta\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n-1)\theta$; $2\cos\frac{1}{2}\theta\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n+1)\theta$
24. $\frac{1}{2}-(n+1)/(n+2)!$; $\frac{1}{2}$ 25. $x^{-2}-(x+n)^{-2}$; x^{-2}
26. $\frac{1}{2}-1/(n+2)!$; $\frac{1}{2}$ 27. $(a+d_1)\dots(a+d_n)/(d_1d_2\dots d_n)$
29. $1-1/(3\,.\,5\dots 2n+1)$

Page 221 EXERCISE Xd

1. 10 24 44 70 102 140 ; 14 20 26 32 38 ; 6 6 6 6.
2. r^2+r+1 3. r^3-r+1 4. $r(r+3)$ 5. $(r-2)(r+3)$
6. $\frac{1}{3}n(n^2+3n+5)$ 7. r^2-r+1 ; $\frac{1}{3}n(n^2+2)$
8. r^3-2r ; $\frac{1}{4}n(n+1)(n^2+n-4)$ 9. $2^{r-1}+2$; 2^n+2n-1
10. $\frac{1}{9}(10^r-1)$; $\frac{1}{81}(10^{n+1}-9n-10)$
11. $\Delta^3 u_r = 16\,.\,3^{r-1}$; $\Delta^4 u_r = 32\,.\,3^{r-1}$ 12. r^3-2r+1
13. $\frac{1}{24}(r^4-10r^3+35r^2-26r+24)$ 14. $2r^2-r+2$

15. $r^3 - r^2$

16. $\frac{1}{2}n(n+1)(n^3 - n - 1)$

17. $4r^2 - 1$; $\frac{1}{3}n(4n^3 + 6n - 1)$

18. $2r^3 - r^2$; $\frac{1}{6}n(n+1)(3n^2 + n - 1)$

19. $3 \,.\, 2^{r-1} - 2$; $3 \,.\, 2^n - 2n - 3$

20. $3 \,.\, 2^{r-1} + 2r - 1$; $3 \,.\, 2^n + n^2 - 3$

21. $(-1)^n/(n+1)$

22. $(-1)^n/(n+2)$

23. $(r-1)(r-2)(r+3)$

24. $\frac{1}{6}(r+1)(r^2 - 4r + 12)$

25. $\frac{1}{15}n(n-1)(n-2)(3n^2 - 6n + 1)$

26. $r(r+3)(2r+1)$; $\frac{1}{6}n(n+1)(3n^2 + 17n + 16)$

27. $\frac{1}{6}r(r^2 - 3r + 8)$; $\frac{1}{24}n(n+1)(n^2 - 3n + 14)$

28. $2 \,.\, 3^{r-1} + r^2$; $3^n + \frac{1}{6}(n-1)(2n^2 + 5n + 6)$

29. $7 \,.\, 2^{r-1} + r^3 - r^2$; $7(2^n - 1) + \frac{1}{12}n(n^2 - 1)(3n + 2)$.

Page 223 EXERCISE Xe

1. 8640

5. $\frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7) + 56\}$

6. $n/(3n+1)$

7. $3^{n+1} - \frac{1}{2}(5n^2 + 9n + 6)$

11. 360

12. $1/\{(n+1)(n+2)\}$

14. $1 - 2^{-n}(n+1)^{-1}$; 1

16. $\frac{1}{3}n(48n^3 + 80n^2 - 6n - 47)$

17. $\frac{1}{6}\{\frac{5}{6} - 1/(n+1) - 1/(n+2) + 2/(n+3)\}$

18. $\frac{1}{3}n(n^2 - 9n + 32)$

21. $(1-a)(2-a) \ldots (n-1-a)/(n-1)!$

22. $n(n+1)/(4n+8)$

28. $\dfrac{a+d}{a-1}\left\{1 - \dfrac{(1+d)(1+2d)\ldots(1+nd)}{(a+d)(a+2d)\ldots(a+nd)}\right\}$

Page 233 EXERCISE XIa

1. $-2, 1$; $\{3 - x - (2n+3)x^n + (2n+1)x^{n+1}\}(1-x)^{-2}$

2. $[1 - 2x + (-1)^{n-1}\{(3n+1)x^n + (3n-2)x^{n+1}\}](1+x)^{-2}$

3. $\{1 + (-1)^{n-1}(2^{n+1} - 1)x^n + (-1)^{n-1}2(2^n - 1)x^{n+1}\}$
$\times (1 + 3x + 2x^2)^{-1}$

4. $A3^r + B(-4)^r$

5. $(A + Br)5^r$

6. $u_r - 3u_{r-1} + 3u_{r-2} - u_{r-3} = 0$

7. $u_r - 10u_{r-1} + 21u_{r-2} = 0$

8. $17 - (-4)^r$

9. $(5 - 2r)3^r$

10. $(r+2)a$

11. $u_r = A + B(-1)^r + C2^r$

12. $3^r - 2^r + 1$

13. $(3 - 5r)2^r + (-4)^r$

14. $(6 \cos r\alpha + \frac{7}{4} \sin r\alpha)5^{r-2}$, $\cos\alpha : \sin\alpha : 1 = 3 : 4 : 5$

15. $r!-1$ 16. $\sum_{1}^{r} k!$ 17. $-2, 1$; 13

18. $\{1-2x-\frac{1}{2}(3^n+1)x^n+\frac{3}{2}(3^{n-1}+1)x^{n+1}\}(1-4x+3x^2)^{-1}$

19. $\{1-x-(2 . 3^n-2^n)x^n+(4 . 3^n-3 . 2^n)x^{n+1}\}(1-5x+6x^2)^{-1}$

20. $u_r=A4^r+B2^r$ 21. $u_r=A(-1)^r+B(-2)^r$

22. $u_r-2u_{r-1}+u_{r-2}=0$ 23. $u_r-5u_{r-1}+4u_{r-2}=0$

24. $2u_r+u_{r-1}-u_{r-2}=0$ 25. $(r-2)(-1)^r$

26. 2^r+2^{2-r} 27. $u_r=A+B2^r+C(-3)^r$

28. $2 . 3^{r-1}+(-4)^{r-1}-10.$ 29. $2^{r/2}\sin\frac{1}{4}r\pi$

30. $\left(\sum_{1}^{r} k!\right)\Big/ r!$ 31. $-3, 3, -1$; 31

32. $\{1+x^2-(n^2+n+1)x^n+2n^2x^{n+1}-(n^2-n+1)x^{n+2}\}(1-x)^{-3}$

33. $u_r-12u_{r-1}+36u_{r-2}=0$

34. $\beta\{(\alpha^{r-1}-\beta^{r-1})-(\alpha^{r-2}-\beta^{r-2})\}/(\alpha-\beta)$

35. $u_r=(A+Br+Cr^2)3^r$ 36. $u_r=A+Br+C3^r$

37. $r^3-7(r-1)^2$ 38. $\{1-\sum_{2}^{r} 1/k!\}(r!)$ 39. $(r+3)!/24$

40. (i) $r+1$, (ii) $(r+1)\sum_{1}^{r-1} k!$ if $r>1$.

Page 240 EXERCISE XIb

1. $12(4^{r-2}-3^{r-2})$; $4^n-2 . 3^n+1$

2. $\frac{1}{3}+\frac{1}{6}4^r$; $\frac{1}{3}n+\frac{2}{9}(4^n-1)$ 3. $3 . 2^{r-2}-1$; $3 . 2^{n-1}-\frac{3}{2}-n$

4. $(1-2x)/(1-4x+3x^2)$; $\frac{1}{2}(3^r+1)$

5. $(2+x)(1-x)^{-2}$; $3r+2$

6. $(1-4x)/(1+4x+4x^2)$; $(3r+1)(-2)^r$

7. $(1+2x)/(1-x+x^2)$

8. $\{2+3x+\frac{2}{3}(2^{2-n}-7 . 2^n)x^n+\frac{1}{3}(7 . 2^n-2^{4-n})x^{n+1}\}$
$\times(1-\frac{5}{2}x+x^2)^{-1}$

9. $(1+x\sqrt{2})/(1-x\sqrt{2}+x^2)$; $\cos\frac{1}{4}r\pi+3\sin\frac{1}{4}r\pi$

10. $\sin\theta/(1-2x\cos\theta+x^2)$; $\sin(r+1)\theta$

11. $6(2^{r-2}+3^{r-2})$; $3 . 2^n+3^n-4$

12. $3+(-2)^{r-1}$; $3n+\frac{1}{3}\{1-(-2)^n\}$

13. $(1+2^{r-1})2^{2r-3}$; $\frac{1}{42}(7 . 4^n+3 . 8^n-10)$

14. $(2-6x)/(1-6x+8x^2)$; 2^r+4^r

15. $(5-16x)/(1-7x+10x^2)$; $3 . 5^r+2^{r+1}$

16. $(5-7x)/(1-x-2x^2)$; $2^r+4(-1)^r$ 17. $(1-3x+x^2+x^3)^{-1}$

18. $\{2-8x-(5^n+3^n)x^n+(3 . 5^n+5 . 3^n)x^{n+1}\}(1-8x+15x^2)^{-1}$

19. $(1+x)/(1-x+\frac{1}{2}x^2)$; $(\cos\frac{1}{4}r\pi+3\sin\frac{1}{4}r\pi)/\sqrt{2^r}$

20. $\{1+(k+2)x+4(k-1)x^2\}\{1+kx+(2k-7)x^2+(6-7k)x^3\}^{-1}$

21. $(4-20x+22x^2)/(1-6x+11x^2-6x^3)$

22. $(4-9x-7x^2)/(1-3x^2-2x^3)$

23. $(8-16x)/(1-\frac{1}{2}x+\frac{1}{4}x^2)$; $2^{4-r}\{\sin\frac{1}{3}(r+1)\pi-4\sin\frac{1}{3}r\pi\}/\sqrt{3}$

24. $(1-x)\cos\theta/(1-2x\cos 2\theta+x^2)$; $\cos(2r+1)\theta$

25. $1+\sum(\cos\frac{2}{3}r\pi-\frac{1}{3}\sqrt{3}\sin\frac{2}{3}r\pi)x^r$.

Page 248 EXERCISE XIc

1. $2+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3}$ 2. $2+\sqrt{5}$ 3. 11th red, 8th blue

7. $9n+2$, $11n+2$ 8. 56, 9 9. 16

11. $\{2 . 3^{r+1}+3(-2)^{r+1}\}\div\{3^{r+1}-(-2)^{r+1}\}$ 12. r^{-1}

13. $1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{2}$ 14. $3+\sqrt{11}$

17. $15-17n$, $13n-11$ 18. 4 19. $(2^r-1)^{-1}$

20. $2r/(1+r)$ 21. $4+\sqrt{19}$

22. $3+\frac{1}{7+}\frac{1}{16+}\frac{1}{11}$; 3, $\frac{22}{7}$, $\frac{355}{113}$

25. 9, 2, 1 ; 2, 6, 1 ; 6, 1, 2. 28. $\frac{1}{4}\{3+(-3)^{1-r}\}$

29. $\{a(-1)^{r-1}+a^{r+1}\}\div\{(-1)^r+a^{r+1}\}$.

Page 250 EXERCISE XId

1. $2^{2r+1}-5^r$ 2. $5^{r-1}-2 . 3^{r-1}$; $\frac{1}{4}(5^n-4 . 3^n+3)$

3. $b(a^r-1)/(a-1)$; br

4. $(4+5x)\div\{(1+2x)(1-x)\}$; $3+(-2)^r$ 5. $(r+2)!\div 2$

7. $u_r=A2^r+B3^r+\frac{1}{2}a$; $u_r=A2^r+B3^r+\frac{1}{2}ar+\frac{7}{4}a$

11. $\frac{1}{2}(2^r+3^{r-1}-5)$; $\frac{1}{4}(2^{n+2}+3^n-10n-5)$

12. $(1-2x)/(1-3x)^2$; $(r+3)3^{r-1}$

13. $(1-6x+7x^2)\div\{(1-x)(1-2x)(1-3x)\}$; $1+2^r-3^r$

14. $\frac{2}{3}(\sqrt{3})\sin\frac{2}{3}r\pi$

19. (i) $(r+1)!$ (ii) $(r+1)!\{1+\frac{1}{2}+\ldots+1/(r+1)\}$

20. $(a-b)(1-abx^2)(1-ax)^{-2}(1-bx)^{-2}$

21. $\frac{1}{6}n(n-1)(4n-5)a-n(n-2)u_1+nu_2+n(n-1)u_3$

22. $u_r=au_{r-1}-\frac{1}{2}(a^2-b^2)u_{r-2}+\frac{1}{6}(a^3-3ab^2+2c^3)u_{r-3}$

23. $u_r=Aa^r+B(-a)^r+$
$\{\cos br-a^2\cos b(r+2)\}(1-2a^2\cos 2b+a^4)^{-1}$

24. $u_r=(r^2+Ar+B)2^{r-1}$

29. $\frac{1}{1}+\frac{1^2}{2}+\frac{3^2}{2}+\ldots\frac{(2n-3)^2}{2}$

Page 258 EXERCISE XIIa

1. 2, 4

4. $(x+y)(x+\omega y)(x+\omega^2 y)$

5. $-1\pm i$, $1\pm\sqrt{2}$

6. $(x^2+y^2)^n$

9. 1, $\frac{1}{4}\{c-1\pm i\sqrt{(10+2c)}\}$ where $c=\pm\sqrt{5}$; ϵ^2, ϵ^{-1}, ϵ^{-2}.

10. abc, $-\sum ab$, $\sum a$.

11. -1, $\frac{1}{2}(1\pm i\sqrt{3})$

13. $(x+y-z)(x+\omega y-\omega^2 z)(x+\omega^2 y-\omega z)$

14. 2, $-\frac{1}{2}$, ω, ω^2

15. $z^2-2z(c^2-d^2)+(c^2+d^2)^2=0$; $z^2-4cdz+(c^2+d^2)^2=0$

18. $(d^2-4abd+4b^2c=0,\ b\neq 0)$ or $(b=d=0,\ a^2\geqslant c)$

19. $\frac{1}{3}n$ not integral

20. $abc\sum a$, $abc-\sum a\sum bc$, $\sum(a^2+bc)$.

Page 263 EXERCISE XIIb

1. x^2-3x+2.

2. $2x^2-5x+2$

3. $(2x-3)^2(x^2+3x-2)$

4. $1-\sqrt{2}$, $\frac{1}{2}(-1\pm\sqrt{5})$

5. $G^2=4H^3$

8. $x-4$.

9. $2x+5$

10. x^2+x+1

11. $3x^2+5x+7$

12. $\pm\sqrt{2}\pm\sqrt{5}$, $-1\pm\sqrt{2}$

13. $\frac{1}{2}$, $\frac{1}{2}$, $-1\pm\sqrt{5}$

14. $n-2$ divisible by 3

16. $c(a-b)(a+b+c+1)=0$

17. $r(p^3+r)=0$

19. 1, 1, $-1\pm i\sqrt{2}$, if $c=1$; -1, -1, $1\pm i\sqrt{2}$, if $c=-1$;
$-i$, $-i$, $i\pm\sqrt{2}$, if $c=i$; i, i, $-i\pm\sqrt{2}$, if $c=-i$.

Page 268 **EXERCISE XIIc**

1. $\frac{1}{13}(1-2x)+(x^2+2x+2)P,\ \frac{1}{13}(6-5x+2x^2)-(x^3+1)P.$

   In 1, 2 P is any polynomial.

2. $\frac{1}{5}(x-2)+(x^2+2x-3)P,\ \frac{1}{5}(-x^2+4x-3)-(x^3-8x+2)P.$

3. $12-29n,\ -7+17n,\ n$ integral. 5. $r:s$ constant.

6. $\frac{1}{81}(4x-11),\ \frac{1}{81}(-8x^2+22x+16)$

7. $x^3,\ x^2(-x^3-x^2+x+1)$

8. $\frac{1}{312}(-x^2+7x+47),\ \frac{1}{312}(x-11)$ 9. $53,\ -41.$

Page 275 **EXERCISE XIId**

1. $5+7(x-2)+4(x-2)^2+(x-2)^3$

2. $-\frac{1}{4}-(2x+1)+\frac{11}{4}(2x+1)^2-2(2x+1)^3+\frac{1}{2}(2x+1)^4$

3. $1-2x+(4+3x)(1-x+x^2)-x(1-x+x^2)^2.$

4. $1+\dfrac{3}{2(x-1)}+\dfrac{5}{2(x-3)}$ 5. $1+\left(\dfrac{a^2}{x-a}-\dfrac{b^2}{x-b}\right)\Big/(a-b)$

6. $\dfrac{3}{2(x+1)}+\dfrac{x+5}{2(x^2+1)}$ 7. $\dfrac{1}{x-1}+\dfrac{1}{2(x+1)}-\dfrac{3x+1}{2(x^2+1)}$

8. $\left(\dfrac{a}{x-a}-\dfrac{ax-c}{x^2+bx+c}\right)\div(a^2+ab+c)$

9. $\dfrac{x}{x^2+1}-\dfrac{x-1}{x^2+x+1}$ 10. $\dfrac{3}{2(1-2x)}+\dfrac{1}{2(1+2x)}-\dfrac{1-2x}{1+4x^2}$

11. $1+4(x+2)-3(x+2)^2+(x+2)^3$

12. $1+4(x-1)+6(x-1)^2+4(x-1)^3+(x-1)^4$

13. $4+2x-z(4+10x)-z^2(5-12x)+z^3(6+x)$

14. $\left(\dfrac{a}{x-a}-\dfrac{b}{x-b}\right)\div(a-b)$ 15. $\dfrac{1}{2(x-1)}+\dfrac{x-1}{2(x^2+1)}$

16. $x+\dfrac{2}{3(x-1)}+\dfrac{x+2}{3(x^2+x+1)}$

17. $1+\left\{\dfrac{a^3}{x-a}+\dfrac{b^2(ax-b^2)}{x^2+b^2}\right\}\div(a^2+b^2)$

18. $\dfrac{x+4}{x^2+2x+2}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x-1)}$

19. $1+\sum\{a^3/(a-b)(a-c)(x-a)\}$

20. $x+a+b+\left(\dfrac{a^3}{x-a}-\dfrac{b^3}{x-b}\right)\div(a-b)$

21. $\frac{1}{8}(\sqrt{2}-2)\left(\dfrac{1}{x+1-\sqrt{2}}+\dfrac{1}{x-1+\sqrt{2}}\right)$

$-\frac{1}{8}(\sqrt{2}+2)\left(\dfrac{1}{x+1+\sqrt{2}}+\dfrac{1}{x-1-\sqrt{2}}\right)$

22. $\dfrac{1}{x-1}-\dfrac{1}{x+1}-\dfrac{2}{x^2+1}+\dfrac{x\sqrt{2}-2}{x^2-x\sqrt{2}+1}-\dfrac{x\sqrt{2}+2}{x^2+x\sqrt{2}+1}$

Page 279 EXERCISE XIIe

1. $(x-1)^{-1}+(x-1)^{-2}+(x-1)^{-3}$
2. $\frac{1}{2}(x-1)(1+x^2)^{-1}-\frac{1}{2}(x-1)^{-1}+(x-1)^{-2}$
3. $28(x-3)^{-1}-28(x-2)^{-1}-26(x-2)^{-2}-15(x-2)^{-3}$
4. $(x-1)^{-1}-(x+1)(x^2+1)^{-1}-(2x+1)(x^2+1)^{-2}$
5. $\frac{1}{3}(4x-1)(x^2-x+1)^{-1}-\frac{4}{3}(x+1)^{-1}-\frac{4}{3}(x+1)^{-2}$
6. $2(x+1)^{-1}+(x+1)^{-2}-(2x-1)(x^2+1)^{-1}-2x(x^2+1)^{-2}$
7. $5\,.\,2^{-r-2}\{1+(-1)^r\}$, $|x|<2$
8. r even, $\frac{6}{5}\{2^r-(-1)^{r/2}\}$; r odd, $\frac{3}{5}\{2^{r+1}+(-1)^{(r-1)/2}\}$; $|x|<\frac{1}{2}$
9. r even, $r+2+(-1)^{r/2}$; r odd, $-r-2+(-1)^{(r+1)/2}$; $|x|<1$
10. $\frac{1}{4}xz+\frac{1}{2}xz^2+(x+1)z^3-\frac{1}{4}(x+2)^{-1}$ where $z=1/(x^2+2x+2)$
11. $(x-1)^{-1}-x^{-5}-x^{-4}-x^{-3}-x^{-2}-x^{-1}$
12. $(x-2)^{-1}+(x-1)^{-1}+(x-1)^{-2}$
13. $\frac{22}{25}(x-2)^{-1}+\frac{3}{5}(x-2)^{-2}-\frac{3}{4}(x-1)^{-1}-\frac{11}{100}(x+3)^{-1}$
14. $2(x^2+1)^{-1}+(x^2+1)^{-2}-2x^{-2}+x^{-4}$
15. $\frac{13}{8}(x-1)^{-1}+\frac{5}{4}(x-1)^{-2}+\frac{1}{2}(x-1)^{-3}-2(x-2)^{-1}+\frac{3}{8}(x-3)^{-1}$
16. $3(x+1)^{-1}+5(x-1)^{-1}+2(x-1)^{-2}+4x(x^2+x+1)^{-1}$
17. $+1$, -1, 0 for $r=3n$, $3n+1$, $3n+2$, n integral; $|x|<1$.
18. $2r+3-2^{r+1}$, $|x|<\frac{1}{2}$
19. $10(x-1)^{-1}+4(x-1)^{-2}+(x-1)^{-3}-10(x-2)^{-1}+6(x-2)^{-2}$
    $-3(x-2)^{-3}+(x-2)^{-4}$
20. $(x^2+4)^{-1}-\frac{7}{4}(x-1)^{-1}+\frac{5}{4}(x-1)^{-2}+\frac{7}{4}(x+1)^{-1}+\frac{5}{4}(x+1)^{-2}$
21. $(x^2+1)^{-1}-(4x+1)(x^2+1)^{-2}-(x^2+2)^{-1}+(4x+2)(x^2+2)^{-2}$.
22. $\frac{1}{12}(r+1)(r+5)+\frac{1}{8}(-1)^r+\frac{1}{8}$, increased by $\frac{1}{3}$ if r divisible by 3.

24. $\dfrac{1}{q^3r(a-x)}+\dfrac{1}{p^3r(b-x)}+\dfrac{1}{pq(c-x)^3}+\dfrac{p-q}{p^2q^2(c-x)^2}$
$+\dfrac{p^2-pq+q^2}{p^3q^3(c-x)}$, $p=b-c$, $q=c-a$, $r=a-b$.

25. $x+\sum\alpha_r+\sum\{\alpha_1{}^{n+1}/\ (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)(x-\alpha_1)\}$

Page 281 **EXERCISE XIIf**

1. $x^2+x+1,\ -x^2-3$

4. $1-\left\{\dfrac{a^3}{x+a}+\dfrac{b^2(ax+b^2)}{x^2+b^2}\right\}(a^2+b^2)^{-1}$

5. $6x-(x-1)^{-1}-(x+1)^{-1}+16(x-2)^{-1}+16(x+2)^{-1}$

6. $2(1+x)^{-1}+3(1+x)^{-2}+4(1-2x)^{-1}$

7. $\frac{1}{2}3^{n-2}\{18(2-x)^{-3}-6n(2-x)^{-2}+n(n-1)(2-x)^{-1}\}$

10. $bc(a+b+c)=0$ 11. $ac=b^2,\ bd=c^2$

12. $-1-\sqrt{3},\ \frac{1}{2}(7\pm\sqrt{37})$

13. $1+\dfrac{(a-c)(b-c)}{(c-d)(x-c)}-\dfrac{(a-d)(b-d)}{(c-d)(x-d)}$

14. $z+(a-2c)z^2+(c^2-ac+b)z^3,\ z=1/(x+c)$

15. $(x+1)(x^2+1)^{-1}+(x-1)(x^2+1)^{-2}-(x-1)^{-1}+(x-1)^{-3}$

16. $8(x^{-1}+x^{-2}+x^{-3})-9(x-1)^{-1}+2(x-1)^{-2}-(x+1)^{-1}$
$+2(x-1)(x^2+1)^{-1}$

17. $k(x-a)^{-2}-2k^2(a-b)(x-a)^{-1}$
$+k^2\{2(a-b)x+a^2-4ab+4b^2-c\}(x^2-2bx+c)^{-1}$,
$k=(a^2-2ab+c)^{-1}$.

Page 290 **EXERCISE XIIIa**

1. 3 roots ; $-2,\ -1$; $0,\ 1$; $1,\ 2$

2. 2 roots ; $0,\ 1$; $1,\ 2$ 4. $-9<k<23$

6. $(x-a_1)\ldots(x-a_{2r-1})+b^2(x-a_2)\ldots(x-a_{2r})=0$
has r unequal roots if every $a_r>a_{r+1}$

11. 3 roots ; $-3,\ -2$; $0,\ 1$; $2,\ 3$

12. 2 roots ; $-1,\ 0$; $2,\ 3$ 20. 2 roots ; $0,\ 1$; $1,\ 2$

21. 2 roots ; $1,\ 2$; $3,\ 4$

Page 295 **EXERCISE XIIIb**

2. 2, 0 3. 0, 1 4. 1, 1 5. n even, 0, 0 ; n odd, 0, 1

9. Between $-2, -1$; 0, 1 ; 1, 2 10. 2, 1

11. 0, 2 12. 1, 0 13. 2, 0

14. $n=3$, roots 1, 1, $-\frac{1}{2}$; $n \geqslant 5$, 2 positive, 1 negative

17. $\{b/(n-1)\}^{n-1} <, > (a/n)^n$. 25. Between 0, 1 and 3, 4.

Page 301 **EXERCISE XIIIc**

1. $x^3+9px-27q=0$, $x(qx+p)^2=q$
2. $a_2^2-2a_1a_3$, $9a_3-a_1a_2$
3. $a_1a_3-4a_4$, $a_1^2a_2-2a_2^2-a_1a_3+4a_4$
4. $(n-1)a_1^2-2na_2$, a_1a_{n-1}/a_n-n, $a_1-a_{n-1}(a_1^2-2a_2)/a_n$
5. $144(a_2^2-a_1a_3)$ 6. $a_3-a_1a_2$, $2a_1^2-6a_2$, a_1a_2/a_3-3
7. $a_1a_3-4a_4$, $2a_4-2a_1a_3+a_2^2$
8. $x(x+a_1)^2+a_2x+a_1a_2-a_3=0$ 9. $(x+2q)^3+p^3(x+3q)=0$
10. $32(a_3-3a_1a_2+2a_1^3)$ 11. $9a_1a_2-27a_3-2a_1^3$
12. $q^2(x+1)^3+p^3(x+2)=0$, $q^2(x^2+x+1)^3+p^3x^2(x+1)^2=0$
13. $a_3^2+a_1^2a_4-4a_2a_4$,
$x^3-a_2x^2+(a_1a_3-4a_4)x-a_3^2-a_1^2a_4+4a_2a_4=0$
14. $3a_1a_4-a_2a_3-5a_5$
15. $x^3+(3a_2-a_1^2)x^2+a_2(3a_2-a_1^2)x+a_2^3-a_1^3a_3=0$,
$x^3+(2a_2-a_1^2)x^2+(4a_1^2a_2-8a_1a_3-a_1^4)x$
$+2(a_1^3-2a_1a_2+2a_3)^2-a_1^4(a_1^2-2a_2)=0$
16. $16(3a_2a_3-2a_1a_4-4a_1^2a_3)$, $16(6a_1a_2a_3-a_1^2a_4-a_3^2)$.

Page 305 **EXERCISE XIIId**

1. $3x^3-9x^2+6x+2=0$
10. $\frac{3}{2}(n-2)(n-3)a_1a_2-\frac{1}{2}(n-1)(n-2)a_1^3-\frac{3}{2}(n^2-9n+2)a_3$
11. $ns_6-6s_1s_5+15s_2s_4-10s_3^2$ 13. $a_3^2-2a_2a_4+2a_1a_5-2a_6$
14. $(n-1)a_1^4-4na_1^2a_2+4(n-3)a_1a_3+2(n+6)a_2^2-4na_4$

Page 310 **EXERCISE XIIIe**

1. -4, $-\omega-3\omega^2$, $-\omega^2-3\omega$
2. $\sqrt[3]{4}-\sqrt[3]{2}$, $\omega\sqrt[3]{4}-\omega^2\sqrt[3]{2}$, $\omega^2\sqrt[3]{4}-\omega\sqrt[3]{2}$ 3. -7, 11, 11

4. $-4,\ -\omega-2\omega^2,\ -\omega^2-2\omega$ 5. $6\cos\frac{1}{9}r\pi,\ r=1,\ 5,\ 7$

6. $27b^2 < 4a^3$ 7. $\sqrt[3]{12}-\sqrt[3]{18}$ 10. $3,\ \omega+2\omega^2,\ \omega^2+2\omega$

11. $\sqrt[3]{3}+2\sqrt[3]{9},\ \omega\sqrt[3]{3}+2\omega^2\sqrt[3]{9},\ \omega^2\sqrt[3]{3}+2\omega\sqrt[3]{9}$

12. $-5,\ -2\omega-3\omega^2,\ -2\omega^2-3\omega$

13. $-1-\frac{1}{2}\sqrt[3]{4}-\sqrt[3]{2},\ -1-\frac{1}{2}\omega\sqrt[3]{4}-\omega^2\sqrt[3]{2},\ -1-\frac{1}{2}\omega^2\sqrt[3]{4}-\omega\sqrt[3]{2}$

14. $-1+4\cos\frac{2}{9}r\pi,\ r=1,\ 2,\ 4$ 16. $5-\sqrt[3]{12}-\sqrt[3]{18}$

19. $a\omega^r+b\omega^s;\ r,\ s=0,\ 1,\ 2$ 21. $1+\sqrt[3]{2}-\sqrt[3]{4}$ 23. $y=x+a_1$

25. $y^3+6qy^2+9q^2y+4q^3+27r^2=0.$
(i) $4q^3+27r^2>0$, (ii) $4q^3+27r^2=0$.

Page 317 **EXERCISE XIIIf**

1. $1\pm\sqrt{2},\ -1\pm 2i$ 2. $\frac{1}{2}(-3\pm\sqrt{5}),\ \frac{1}{2}(-1\pm\sqrt{17})$

3. $3\pm2\sqrt{2},\ 2\pm\sqrt{3}$ 4. $\pm1,\ -3,\ -\frac{1}{3},\ \frac{1}{2}(3\pm\sqrt{5})$

5. $\frac{1}{2}(3\pm\sqrt{21}),\ \frac{1}{2}(-1\pm\sqrt{13})$ 8. $256(I^3-27J^2)$

9. $I^3<27G^4$, 2 x-axal and 2 conjugate; $I^3=27G^4\neq0$, 2 equal x-axal and 2 conjugate; $I=G=0$, 4 zero; $I^3>27G^4$, two pairs of conjugate.

10. (i) includes $(H=0,\ I>0)$; (v) includes $(H=0, I<0)$ and $(H\neq0,\ I=0)$.

11. $-2\pm\sqrt{7},\ 2\pm\sqrt{3}$ 12. $-3\pm\sqrt{7},\ 1\pm i\sqrt{3}$

13. $1,\ 2,\ \frac{1}{2},\ 2\pm\sqrt{3}$

14. $1,\ \frac{1}{4}\{-1+c\pm i\sqrt{(10+2c)}\}$ where $c=\pm\sqrt{5}$

15. $y^5-6y^4+7y^3+7y^2-6y+1=0;\ -2,\ 1\pm\sqrt{3},\ \frac{1}{2}(1\pm\sqrt{5})$

16. $(p-n)(p-n+m)^2=(2p-n)^2$ 17. $x=2y$

18. $G^2+4H^3\neq0$, 2 x-axal and 2 conjugate; $G^2+4H^3=0$, $G\neq0$, 3 equal x-axal and another x-axal; $G=H=0$, 4 zero.

19. $2,\ 2,\ \frac{1}{2},\ \frac{1}{2},\ -1,\ -1$

20. $(x-1)^2+3(x+2)^2,\ 2(x-1)^2-(x+2)^2$

21. $2\cos\frac{1}{15}m\pi,\ m=2,\ 8,\ 14,\ 20,\ 26$

22. $(a_0z-2a_2)^3-4I(a_0z-2a_2)+16J=0$

23. $\alpha+\beta=\gamma+\delta;\ \alpha=\beta.$

Page 320 EXERCISE XIIIg

1. $2a^3 - 9ab + 27c = 0$; $b^3 = a^3c$ 3. $cq^2 - bqr + r^2$
6. (i) $(36a_1a_2a_3 - 27a_2^3 - 8a_3^2)/a_3^2$ (ii) $3(a_3^2 - 9a_1a_2a_3 + 9a_2^3)$
8. $q > p^4$ 9. Also one root if $q(81q^7 + 125r^2) > 0$.
10. + + − 17. $p^2(b^2 - ac) = a^2(q^2 - pr)$ 18. + −
20. Between a, b; b, c. Also $< a$ if $a + b + c + d > 0$, or $> c$ if $a + b + c + d < 0$
21. $-\sqrt[3]{5} - \sqrt[3]{25}$, $-\omega\sqrt[3]{5} - \omega^2\sqrt[3]{25}$, $-\omega^2\sqrt[3]{5} - \omega\sqrt[3]{25}$
22. One negative 23. + −
25. $48(a_1^2 - a_2)$; $36a_2^2 - 32a_1a_3 + 2a_4$; $16(3a_2a_3 - 2a_1a_4 - 4a_1^2a_3)$
26. $r^2 - pqr + p^2s = 0$; $(px^2 + r)(rx^2 + prx + ps) = 0$
27. $(\alpha - 3)/(\alpha + 1)$, $(\alpha + 3)/(1 - \alpha)$

Page 331 EXERCISE XIVa

1. 19 2. 1446 5. C, mon. decr., 5
6. Osc., ± 1 7. D, mon. incr. 8. Osc. infin.
9. C, mon. incr., $\frac{1}{2}\pi$ 10. Osc., ± 1
11. (5) 12, 5, $x = 5$. (9) $\frac{1}{2}\pi$, $\frac{1}{4}\pi$, $x = \frac{1}{2}\pi$. (10) $\frac{5}{4}$, $-\frac{1}{2}$, $x = 0, \pm 1$
13. $\frac{1}{4}\sqrt{(28 + 41/\epsilon)}$ 14. C, mon. decr., 0 15. Osc., 1, 0
16. D, mon. incr. 17. Undefined for $n = 4m + 2$
18. C, mon. decr., 1 19. Osc., ± 1 20. D 21. C, 0
22. Osc. infin. 23. Osc., ± 1 24. Osc. infin.
25. (20) $m = -1$. (21) sin 1, (sin 5)/$\sqrt{5}$, $x = 0$. (23) ± 1, $x = 0, \pm 1$. (24) $m = -1$, $x = 1$, $-\frac{1}{2}$.
26. $-\frac{1}{2}$ 27. mon. decr., mon. incr.
28. mon. incr., mon. decr.

Page 337 EXERCISE XIVb

18. Osc. fin., C. 20. 1 25. (ii) C

Page 340 EXERCISE XIVc

1. D. 2. C. 3. D. 4. C if $b + c < -1$
5. C. 6. C if $b > 1$ 7. D. 8. D.
11. C if $q < -2$ 12. C if $b > 1$ 13. C. 14. C.
15. D. 16. C. 19. D. 20. C.

Page 352 **EXERCISE XIVd**

1. C. 2. D. 3. C.

4. $x < -1$, osc. infin.; $-1 < x < 1$, C; $x > 1$, D

5. C. 6. Same answers as 4. 7. D unless $c = 1$.

8. $|x| < 1$, C; $|x| > 1$, D 9. C if $a > 1$. 10. D, C, C.

12. $a < c$, D to zero; $a = c$, C; $a > c$, D to $+\infty$

13. D. 14. C. 15. C. 16. C. 17. C if $|x| < 1$

18. C if $|x| < 1$ 19. D. 20. C. 21. D.

22. D to zero if $|x| < 1$ or $x = -1$; C if $|x| > 1$ or $x = 1$.

23. C, C 24. D. 25. C if $p > 2$

28. (i) C to 1, (ii) D to zero.

Page 358 **EXERCISE XIVe**

[* denotes non-existence]

1. D, but undefined if $1/x$ is a negative integer.

2. $|x| < 1$, abs. C; $|x| > 1$ and $x = 1$, D; $x = -1$, undefined.

3. $x < -1$, osc. infin.; $x = -1$, osc. fin.; $|x| < 1$, abs. C; $x > 1$, D.

4. $x < -1$, D; $|x| < 1$, abs. C; $x > 1$, osc. infin.

5. $x(1-x)^{-2}$ 6. *, *, 4, 2 7. 2, 2, 2, 4 8. 4, 4.

10. $|x| \neq 1$, abs. C; $x = -1$, C; $x = 1$, D.

11. $|x| > 1$, abs. C; $-1 < x < 1$, D; $x = -1$, C.

12. $|x| < 1$, abs. C; $x < -1$, osc. infin.; $x = -1$, C; $x > 1$, D.

13. $xy \div \{(1-x)(1-y)\}$ 14. 1, 1, *, 2.

17. $|x| < 1$, abs. C; $x = 1$, abs. C if $\gamma > \alpha + \beta$, D if $\gamma < \alpha + \beta$; $x = -1$, abs. C if $\gamma > \alpha + \beta$, C if $\alpha + \beta - 1 < \gamma < \alpha + \beta$, osc. fin. if $\gamma = \alpha + \beta - 1$, osc. infin. if $\gamma < \alpha + \beta - 1$; $x < -1$, osc. infin.; $x > 1$, D.

18. $-\frac{1}{2}, \frac{1}{2}, e^{-1} - \frac{1}{2}$

Page 362 **EXERCISE XIVf**

[** denotes recurrence]

1. $1 + \frac{1^*}{1+}\,\frac{1^*}{2}$ 2. $3 + \frac{1^*}{3+}\,\frac{1^*}{6}$ 3. $3 + \frac{1^*}{9+}\,\frac{1^*}{6}$

4. $n + \frac{1^{**}}{2n}$ 5. $\frac{1}{2}(1 + \sqrt{13})$ 6. $\frac{1}{4}(9 + \sqrt{5})$

7. $2-\frac{1}{5}\sqrt{15}$ 8. $\frac{1}{2}\{a+\sqrt{(a^2+4b)}\}$ 9. $5+\frac{1^{**}}{10}$

10. $\frac{1}{1+}\frac{1}{3+}\frac{1^{**}}{4}$ 11. $n+\frac{1^*}{n}\frac{1^*}{+2n+}$

12. $\frac{1}{3+}\frac{1^*}{1}\frac{1}{+1+}\frac{1}{1+}\frac{1^*}{4}$ $-\frac{1}{1+}\frac{1^*}{1}\frac{1}{+1+}\frac{1}{1+}\frac{1^*}{4}$

13. $3+\frac{1^*}{1}\frac{1^*}{+6}$ $\frac{213}{55}$ 14. $4-\sqrt{3}$ 18. 1.

19. $(e-2)/(e-1)$ 20. $-1+\log 4$

23. (iii) $\alpha_1+\alpha_1\alpha_2+\alpha_1\alpha_2\alpha_3+\dots$ $\frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2}+\dots$

Page 365 **EXERCISE XIVg**

1. $q>1$ or $(q=1, p<-1)$ 2. $k<\frac{3}{2}$ 3. $c+d-a-b>1$

4. $-1<x\leqslant 1$; 0 if $|x|<1$, -1 if $x=1$

5. (i) osc. fin., (ii) C. 13. Osc. fin. 14. $p>1$

15. $x^{2k}/(x^{2k}-1)$; 1 if $|x|>1$, 0 if $|x|<1$

16. $1/(x-1)$ 18. 1.